AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LIII, Number 3

JULY, 1931

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

Sufficient conditions in the problem of Lagrange with variable end conditions. By MARSTON MORSE, ... 517

On the problem of Lagrange. By LAWRENCE M. GRAVES, . . 547

Perspective elliptic curves. By ELIZABETH MORGAN COOPER, . . 555

Conjugate nets and the lines of curvature. By ERNEST P. LANE, . 573

On the expansion of harmonic functions in terms of normal-orthogonal harmonic polynomials. By GAYLORD M. MERRIMAN, . 589

On some problems of Tchebycheff. By J. GERONIMUS, . . . 597

Three notes on characteristic exponents and equations of variation in celestial mechanics. By AUREL WINTNER, . . . 605

The equation of stability of periodic orbits of the restricted problem of three bodies in Thiele's regularising coördinates. By JENNY E. ROSENTHAL, ... 626

Algebras of certain doubly transitive groups. By R. D. CARMICHAEL, 631

Cayley diagrams on the anchor ring. By R. P. BAKER, . . . 645

Concerning continuous images of the interval. By G. T. WHYBURN, 670

On quasi-metric spaces. By W. A. WILSON, ... 675

Double implication and beyond. By H. B. SMITH, . . . 685

Regular bilinear transformations of sequences. By P. A. FRALEIGH, 697

On some general commutation fromulas. By NEAL H. MCCOY, . 710

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $7.50 (foreign postage 25 cents); single numbers $2.00.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at The Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

SUFFICIENT CONDITIONS IN THE PROBLEM OF LAGRANGE WITH VARIABLE END CONDITIONS.

By Marston Morse.*

1. *Introduction.* A general formulation of problems of this type has been given by Bolza † who obtained necessary conditions analogous to the Euler equations and the transversality conditions. Bliss ‡ lightened Bolza's hypotheses, and gave a new form to the problem. In its simpler aspects in the plane it was recognized by Hilbert § and others that the conditions analogous to the Jacobi conditions could be given in terms of characteristic roots of an auxiliary boundary value problem. A necessary condition of the latter sort has been recently obtained by Cope.¶ The author obtains a similar necessary condition in somewhat simpler form.

Sufficient conditions in this general problem have never been established. The present paper gives and establishes such conditions.

The author has also studied the complete problem of which the minimum problem is a special case, namely, the problem of finding an extremal which gives to a fundamental quadratic form a prescribed type number. The solution is in terms of characteristic roots of a linear boundary value problem.

Finally the results obtained lead to new types of separation and oscillation theorems involving the relative distribution of characteristic roots of two different auxiliary boundary value problems. This is in contrast with earlier comparison theorems, more geometric in nature involving focal points and conjugate points. See Morse II.** Other theorems of this nature even more general in character will be published shortly by the author.

---

* Presented to the Society, September 12 (1930).

† Bolza, *Mathematische Annalen*, Vol. 74 (1913), p. 430.

‡ Bliss, *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 305. Also see Bliss, *American Journal of Mathematics*, Vol. 52 (1930), p. 674.

§ In this connection see the following:

Lovitt, *Linear Integral Equations*, p. 207, for work of Hilbert; Richardson, *Mathematische Annalen*, Vol. 68 (1910), p. 279; Plancherel, *Bulletin des Sciences Mathématiques*, Vol. 47 (1923), p. 376; Bliss, *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 317.

¶ Cope, University of Chicago, Doctor's Thesis.

** Certain results in the following papers will be used.

Morse I. *Transactions of the American Mathematical Society*, Vol. 31 (1929), p. 379. This paper shows the connections with a theory in the large.

THE ACCESSORY PROBLEM.

2. *The transversality conditions.* In the space of the variables x and

$$(y) = (y_1, \cdots, y_n)$$

let there be given a curve

$$(2.1) \qquad y_i = \bar{y}_i(x) \qquad a^1 \leqq x \leqq a^2 \qquad (i = 1, \cdots, n)$$

of class C'. Points neighboring the initial and final end points of g will be denoted respectively by

$$(2.2) \qquad (x^s, y_1^s, \cdots, y_n^s) = (x^s, y^s) \qquad (s = 1, 2)$$

where $s = 2$ at the final end point and 1 at the initial end point.

We consider curves of class C' neighboring g. Such curves will be called *differentially admissible* if they satisfy m differential equations of the form

$$(2.3) \qquad \phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m < n).$$

We suppose g is differentially admissible, and that along g the functional matrix of the functions (2.3) with respect to the variables y_i' is of rank m.

A curve neighboring g will be said to be *terminally admissible* if its end points are given for some value of (α) by the functions

$$(2.4) \qquad x^s = x^s(\alpha_1, \cdots, \alpha_r) \qquad y_i^s = y_i^s(\alpha_1, \cdots, \alpha_r) \qquad 0 < r \leqq 2n + 2$$

Morse II. *Mathematische Annalen*, Band 103 (1930), p. 52. Here are separation theorems.

Morse III. "Sufficient Conditions in the Problem of Lagrange with Fixed End points," *Annals of Mathematics*, Vol. 32 (1931).

In this paper sufficient conditions are derived, it is believed for the first time, under the following hypotheses regarding normalcy. If the extremal g be defined on the closed interval (ab) of the x axis it shall be normal relative to the Euler conditions on every subinterval of (ab). Previous methods of proof break down because now a set of extremals through a point p on g's extension just before g *need not form a field* near g. Previously it was assumed that g was normal on every sub-interval of an interval including (ab) in its interior. For a definition of normalcy see the following paper.

Morse and Myers, *Proceedings of the American Academy of Arts and Sciences*, Vol. 66 (1931), p. 235. Here the Euler and transversality conditions are derived in forms necessary for the present paper.

Other references follow:

Bolza, *Vorlesungen über Variationsrechung*, hereafter referred to as Bolza; Hadamard, *Leçons sur le calcul des variations*, Vol. 1, Paris, 1910; Carathéodory, "Die Methode der geodätischen Äquidistanten und das Problem von Lagrange," *Acta Mathematica*, Vol. 47 (1926), p. 199; J. Radon, "Über die Oszillationstheoreme der konjugierten Punkte beim Problem von Lagrange," *Sitzungsberichte der mathnaturwissenschaftlichen Abteilung der Bayrischen Akademie der Wissenschaften zu München* (1927), p. 243.

where these functions of (α) are defined for (α) near (0) and reduce to the end points of g for $(\alpha)=(0)$. We assume that the functional matrix of the $2n+2$ functions in (2.4) is of rank r for $(\alpha)=(0)$.

A curve that is both differentially and terminally admissible will be called admissible.

We seek first the conditions under which g affords a minimum for the expression

$$J=\int_{x^1}^{x^2} f(x,y,y')\,dx+G(x^s,y^s) \tag{2.5}$$

among admissible curves of class C'.

The functions f and ϕ_β are to be of class C''' while the functions G and the end point functions in (2.4) need be of no more than class C''.

It is known * that if g affords a minimum for J there exists a constant λ_0 and m functions $\lambda_\beta(x)$ not all identically zero if $\lambda_0=0$, such that g satisfies the equations

$$(d/dx)F_{y_i'}-F_{y_i}=0 \qquad (i=1,\cdots,n) \tag{2.6}$$

where †

$$F=\lambda_0 f+\lambda_\beta\phi_\beta \qquad (\beta=1,\cdots,m)$$

while the following transversality relations hold

$$\lambda_0 dG+[(F-\bar{y}_i'F_{y_i'})dx^s+F_{y_i'}\,dy_i^s]_1^2=0. \tag{2.7}$$

Here dx^s and dy_i^s are the differentials of the functions (2.4) evaluated for $(\alpha)=(0)$, while dG is to be evaluated for (x^1,y^1) and (x^2,y^2) at the respective ends of g. Here and elsewhere $[\ ]_1^2$ shall mean the difference between the value of the bracket evaluated for $s=2$ with (x,y,y') at the final end point of g, and the corresponding evaluation at the initial end point of g.

It will be convenient to set

$$G[x^s(\alpha),y^s(\alpha)]\equiv g(\alpha). \tag{2.8}$$

If we regard (2.7) as an identity in the independent differentials $d\alpha_h$ we obtain the following relations

$$\lambda_0 g_h+[(F-\bar{y}_i'F_{y_i'})x_h^s+F_{y_i'}y_{ih}^s]_1^2=0 \qquad (h=1,\cdots,r;\ i=1,\cdots,n) \tag{2.9}$$

---

* See Morse and Myers, § 4, *loc. cit.*

† The summation convention of tensor analysis is to be used throughout.

where the subscript h indicates differentiation with respect to α_h.

3. *The second variation.* It is convenient to set

$$(3.1)\qquad P_{ij}(x) \equiv F_{y_i y_j} \qquad Q_{ij} \equiv F_{y_i y_j'} \qquad R_{ij} \equiv F_{y_i' y_j'}$$

where the partial derivatives of F are to be evaluated along g. As is conventional we then set

$$(3.2)\qquad 2\omega(\eta, \eta') = P_{ij}\eta_i\eta_j + 2Q_{ij}\eta_i\eta_j' + R_{ij}\eta_i'\eta_j'.$$

A set (α) in (2.4) determines a set of end points. Let us take a set of points (α) of the form

$$(3.3)\qquad \alpha_h = \alpha_h(e) \qquad \alpha_h(0) = 0 \qquad (h = 1, \cdots, r)$$

where the functions $\alpha_h(e)$ are of class C'' for (e) near (0). Suppose that we have a family of admissible curves

$$(3.4)\qquad y_i = y_i(x, e) \qquad (i = 1, \cdots, n)$$

taking on the end points determined by (α) in (3.3). That is, we suppose that we have, subject to (3.3), the following identities in e

$$(3.5)\qquad y_i[x^s(\alpha), e] \equiv y_i^s(\alpha). \qquad (s = 1, 2;\ i = 1, \cdots, n).$$

We suppose that for $e = 0$, (3.4) gives the extremal g.

We shall need * the results of differentiating these identities with respect to e. Upon setting $\alpha_h' = u_h$ a first differentiation gives, $(h = 1, \cdots, r)$,

$$(3.6)\qquad y_{ie} = y_{ih}^s u_h - y_{ix} x_h^s u_h \qquad \alpha_h' = u_h.$$

A second differentiation and evaluation for $e = 0$ yields the result, (s not summed), $(h, k = 1, \cdots, r)$:

$$(3.7)\qquad y_{iee} + 2y_{iex}x_h^s u_h = (y^s{}_{ihk} - \bar{y}_i'' x_h^s x_k^s - \bar{y}_i' x_{hk}^s)u_h u_k + (y_{ih}^s - \bar{y}_i' x_h^s)u_h'.$$

We now suppose J evaluated along the curves (3.4) between the end points determined by (3.3). We have †

$$(3.8)\qquad \lambda_0 dJ/de \equiv \lambda_0 g_h u_h + (Fx_h^s)\; u_h + \int_{x^1}^{x^2} (F_{y_i} y_{ie} + F_{y_i'} y_{ixe})\, dx.$$

---

* What we need for the general theory is to know that the second variation takes the form (3.9). For the general theory the reader need not follow through the algebra except to that end.

† The symbol $(\;)_1^2$ which is used in this place alone, means evaluation at the two ends of the curve $y_i = y_i(x, e)$.

We shall differentiate (3. 8) with respect to e and set $e = 0$. From differentiation of the integral in (3. 8) and appropriate integration by parts we obtain terms of the form

$$A = [(F_{y_i} y_{ie} + F_{y_i'} y_{iex}) x_h^s]_1^2 u_h + [y_{iee} F_{y_i'}]_1^2$$

together with the integral

$$B = 2 \int_{a^1}^{a^2} \omega(\eta, \eta') \, dx \qquad\qquad \eta_i = y_{ie}(x, 0).$$

Differentiation of the remaining terms in the right member of (3. 8) gives terms of the form

$$C = \lambda_0 g_{hk} u_h u_k + [(F_x + \bar{y}_i' F_{y_i} + \bar{y}_i'' F_{y_i'}) x_h^s x_k^s + F x_{hk}^s]_1^2 u_h u_k \\ + [(F_{y_i} y_{ie} + F_{y_i'} y_{ixe}) x_h^s]_1^2 u_h + (\lambda_0 g_h + [F x_h^s]_1^2) u_h'.$$

We wish to reduce the sum $A + C$ to a quadratic form in (u). We first replace y_{ie} in A and C by the right member of (3. 6). We thereby obtain additional quadratic terms of the form

$$D = 2[F_{y_i}(x_h^s y_{ik}^s - \bar{y}_i' x_h^s x_k^s)]_1^2 u_h u_k.$$

The terms remaining in $A + C$ which are not quadratic in (u) have the form

$$[F_{y_i'}(y_{iee} + 2y_{iex} x_h^s u_h)]_1^2 + \lambda_0 g_h u_h' + [F x_h^s]_1^2 u_h'$$

which terms with the aid of (3. 7) and the transversality relations reduce to the form

$$E = [F_{y_i'}(y^s_{ihk} - \bar{y}_i'' x_h^s x_k^s - \bar{y}_i' x_{hk}^s)]_1^2 u_h u_k.$$

Thus we have for $e = 0$, setting $\alpha_h' = u_h$,

$$\lambda_0 d^2 J / de^2 = b_{hk} u_h u_k + 2 \int_{a^1}^{a^2} \omega(\eta, \eta') \, dx \qquad\qquad b_{hk} = b_{kh} \tag{3.9}$$

where $2b_{hk}$ is the sum of the coefficients of $u_h u_k$ and $u_k u_h$ in C, D, and E. We thereby find that

$$b_{hk} = \lambda_0 g_{hk} + [(F_x - \bar{y}_i' F_{y_i}) x_h^s x_k^s \\ + (F - \bar{y}_i' F_{y_i'}) x_{hk}^s + F_{y_i}(x_h^s y_{ik}^s + x_k^s y_{ih}^s) + F_{y_i'} y^s_{ihk}]_1^2. \tag{3.10}$$

For $e = 0$ let us set

$$(\partial / \partial e) x^s(\alpha(e)) = \gamma^s$$

and indicate evaluation of $\eta_i(x)$ at the respective ends of g by the superscript s. We shall prove the following lemma.

LEMMA. *The quadratic form*

$$b_{hk}u_h u_k \qquad (h, k = 1, \cdots, r)$$

is identically equal to a quadratic form in a suitable subset of r of the $2n+2$ variables γ^s, η_i^s.

Let us set (s not summed)

$$y_{ih}^s(0) - x_h^s(0)\bar{y}_i'(a^s) = c_{ih}^s \qquad (i = 1, \cdots, n;\ s = 1, 2). \tag{3.11}$$

From (3.6) we obtain then a fundamental relation

$$\eta_i^s = c_{ih}^s u_h. \tag{3.12}$$

A parallel relation is obtained from (2.4), namely

$$\gamma^s = x_h^s u_h \qquad (\alpha) = (0). \tag{3.13}$$

We need to consider the two matrices

$$\left\| \begin{matrix} c_{ih}^s \\ x_h^s \end{matrix} \right\| = p \qquad \left\| \begin{matrix} y_{ih}^s \\ x_h^s \end{matrix} \right\| = q \tag{3.14}$$

of which the first is the matrix of the coefficients of the variables (u) in the system (3.12) and (3.13), and the second is the functional matrix of the system (2.4), evaluated for $(\alpha) = (0)$.

By hypothesis $\boldsymbol{q}$ is of rank r. It follows that $\boldsymbol{p}$ is of rank r. For $\boldsymbol{p}$ can be readily obtained from $\boldsymbol{q}$ by adding suitable multiples of its last two rows to the preceding rows.

Hence the variables u_h may be expressed as a linear combination of r of the variables γ^s, η_i^s and the lemma follows directly.

4. *The accessory boundary problem.* From this point on we suppose that the extremal g is *normal* relative to the Euler equations and transversality conditions. See Morse and Myers, § 5. The constant λ_0 is not then zero. It can be taken as unity and this choice we suppose made.

A set of functions $\eta_i(x)$ of class C' which satisfy the differential conditions

$$\Phi_\beta(\eta) \equiv \phi_{\beta y_i}\eta_i + \phi_{\beta y'_i}\eta_i' = 0 \qquad (\beta = 1, \cdots, m;\ i = 1, \cdots, n) \tag{4.1}$$

and which with a set of constants (u) satisfy the terminal conditions

$$\eta_i^s = c_{ih}^s u_h \tag{4.2}$$

will be called *admissible*.

The form of the second variation suggests the accessory problem of finding functions $\eta_i(x)$ and constants (u) which give a minimum to*

$$I(\eta, u, \sigma) = b_{hk}u_h u_k + 2\int_{x^1}^{x^2}[\omega(\eta, \eta') - \sigma\eta_i\eta_i]dx$$

for a given σ, relative to all admissible sets (η) and constants (u).

Suppose we have a solution of the accessory minimum problem for a given σ, in the form of n functions $\eta_i(x)$ of class C', and a set of constants (u). It follows from the fact that g is normal as stated that the solution (η) is normal in the accessory minimum problem relative to the new Euler and transversality conditions. For we note that for $\lambda_0 = 0$ the Euler and transversality conditions for the two problems are the same. In particular, upon noting that $x^s = a^s$ in the accessory problem, we find that the transversality conditions (2.9) for both problems, if $\lambda_0 = 0$, may be given the form

$$[\lambda_\beta\phi_{\beta y_i'}c_{ih}{}^s]_1^2 = 0 \qquad (h = 1, \cdots, r).$$

There must then exist m multipliers $\mu_\beta(x)$ of class C' which with the functions $\eta_i(x)$ satisfy the differential equations

$$(4.3) \quad (d/dx)\Omega_{\eta_i'} - \Omega_{\eta_i} = 0, \quad \Phi_\beta = 0 \quad (i = 1, \cdots, n;\ \beta = 1, \cdots, m)$$

where

$$(4.4) \qquad \Omega(\eta, \eta', \mu, \sigma) = \omega(\eta, \eta') + \mu_\beta\Phi_\beta - \sigma\eta_i\eta_i.$$

The functions (η), (μ) and the constants (u) and σ must also satisfy the transversality conditions,

$$(4.5) \qquad b_{hk}u_k + [\Omega_{\eta_i'}c_{ih}{}^s]_1^2 = 0 \qquad (h, k = 1, \cdots, r)$$

and the given boundary conditions

$$(4.6) \qquad \eta_i{}^s = c_{ih}{}^s u_h.$$

The differential equations (4.3) *and the boundary conditions* (4.5) *and* (4.6) *define what will be called the accessory boundary problem.*

By a *solution* of the accessory boundary problem is meant a set of functions $\eta_i(x)$ of class C'' which with multipliers $\mu_\beta(x)$ of class C' and constants (u) and σ satisfy the conditions of the problem. We have the theorem

---

* The form of the integrand is suggested by the idea of dominating the sign of the second variation by new terms. It also comes in if one seeks to minimize the second variation subject to a condition that the integral of $\eta_i\eta_i$ over the given interval be unity. See Cope, *loc. cit.*

THEOREM 1. *A solution (η), (u) of our problem of minimizing $I(\eta, u, \sigma)$ for a fixed σ must possess multipliers (μ) which with (η), (u), and σ, give a solution of the accessory boundary problem.*

By a *characteristic solution* * of the accessory boundary problem will be meant a solution for which $(\eta) \not\equiv (0)$. The corresponding value of σ will be called a *characteristic root.*

5. *The necessary condition on the characteristic roots.* We shall prove the following lemma.

LEMMA. *If (η) is a characteristic solution with constants (u) and σ, $I(\eta, u, \sigma) = 0$.*

Since (η) satisfies $\Phi_\beta = 0$, we have

$$I(\eta, u, \sigma) = b_{hk} u_h u_k + 2 \int_{a^1}^{a^2} \Omega(\eta, \eta', \mu, \sigma)\, dx. \tag{5.1}$$

If we make use of the homogeneity of Ω and integrate by parts in the usual way, we find as a consequence of (4.3) that

$$I(\eta, u, \sigma) = b_{hk} u_h u_k + [\Omega_{\eta_i'} \eta_i^s]_1^2. \tag{5.2}$$

If finally we make use of (4.5) we see that $I(\eta, u, \sigma) = 0$ as was to be proved.

We now come to the following theorem.

THEOREM 2. *If a normal extremal g furnish a minimum for the given problem it is necessary that there exist no characteristic solution of the accessory boundary problem for which $\sigma < 0$.*

Suppose σ_1 were a negative characteristic root and (η) the corresponding characteristic solution with its constants (u). We have

$$I(\eta, u, 0) = I(\eta, u, \sigma_1) + 2 \int_{a^1}^{a^2} \sigma_1 \eta_i \eta_i\, dx. \tag{5.3}$$

By the preceding lemma $I(\eta, u, \sigma_1) = 0$ and since $(\eta) \not\equiv (0)$ we have from (5.3) that

$$I(\eta, u, 0) < 0.$$

We now seek an admissible family of curves of which $I(\eta, u, 0)$ is the second variation.

---

* We shall presently assume that every segment of g is normal relative to the Euler conditions. Under this hypothesis the only solutions $(\eta, \mu) \not\equiv (0, 0)$ will be the solutions for which $(\eta) \not\equiv (0)$.

For the given set (u) let γ^s be defined by (3.13). The sets γ^s, (η), (u) then satisfy (3.12), (3.13) and (4.1). In the normal case * it is known that there will then exist an admissible family of curves of the form $y_i = y_i(x, e)$ and functions $\alpha_h(e)$, such as (3.3) and (3.4), such that γ^s, (η) and (u) are the corresponding variations for this family. But this is impossible if g furnish a minimum since $I(\eta, u, 0) < 0$.

Thus the theorem is proved.

6. *The non-tangency hypothesis.* In ordinary problems involving transversality of a manifold to a given extremal it is generally customary to assume that the manifold is not tangent † to the given extremal, or to insure this by other assumptions. There is here a corresponding assumption apparently hitherto unnoticed. The problem could be treated without this hypothesis, and a summary of results in such a case will be published separately. But sufficient conditions in the case where the hypothesis is made are much simpler than in the case where it is not made. Moreover simple examples in the plane will show the undesirable complexity and unimportance of the special case.

In both cases one is led to an accessory problem involving a parameter σ, but in the more general case the parameter need be introduced only into the integral and not into the end conditions.

In the space of the $2n + 2$ variables (x^s, y_i^s) consider the 2-dimensional manifold defined by the equations, (see 2.1),

$$(6.1) \qquad y_i^s = \bar{y}_i(x^s) \qquad (s = 1, 2;\ i = 1, \cdots, n)$$

for x^s neighboring a^s. This manifold is essentially the arbitrary combination of a point of g near the final end of g with a point of g near the initial end of g. We now regard the equations (2.4) as defining another manifold

$$(6.2) \qquad y_i^s = y_i^s(\alpha) \qquad x^s = x^s(\alpha)$$

in the same $(2n + 2)$-dimensional space. We term (6.1) the *extremal manifold* and (6.2) the *terminal manifold.*

Our non-tangency condition is simply that the extremal manifold and the terminal manifold possess no common tangent line at the point $(\alpha) = (0)$ *on the terminal manifold.*

---

* See Morse and Myers, § 5, Theorem 5.

† In the parametric form this assumption with the assumption that the Weierstrass function $F_1 \neq 0$, enables one to prove the existence of a field of extremals cutting the given manifold transversally.

We state the following lemma.

LEMMA A. *A necessary and sufficient condition for the non-tangency condition to hold is that the matrix* $\| c_{ih}{}^{s} \|$ *of* (3.11) *be of rank* r.

If one writes down as columns the direction numbers of the tangents to the parametric curves of the two manifolds concerned, one verifies the lemma readily. In particular a set of such direction numbers for the terminal manifold are given by the r columns of the matrix

$$\left\| \begin{matrix} x_h{}^s \\ y_{ih}{}^s \end{matrix} \right\| \qquad \begin{matrix} (i = 1, \cdots, n). \\ (h = 1, \cdots, r) \end{matrix}$$

This matrix is of rank r by hypothesis (§ 2). The corresponding matrix for the extremal manifold consists of two columns

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ \bar{y}_i{}' & 0 \\ 0 & \bar{y}_i{}' \end{matrix} \right\| \qquad (i = 1, \cdots, n).$$

The non-tangency condition means that there is no linear relation between the columns of the two matrices which actually involves both matrices. That there is such a relation if and only if $\| c_{ih}{}^{s} \|$ is of rank less than r will be left to the reader to prove.

We shall point out some of the advantages of assuming the non-tangency condition which we shall assume from now on.

The variations (η) and constants (u) appearing in the second variation are related as follows:

$$\eta_i{}^s = c_{ih}{}^s u_h \tag{6.3}$$

as we have seen. In the case of non-tangency we can solve (6.3) for (u) in terms of a suitable subset of r of the variations $\eta_i{}^s$. Instead of the lemma of § 3 we then have the following.

LEMMA B. *In the case of non-tangency the second variation may be written in the form*

$$\lambda_0 d^2J/de^2 = q(\eta) + 2\int_{a_1}^{a_2} \omega(\eta, \eta')\, dx$$

where $q(\eta)$ *is a quadratic form in* r *of the variations* $\eta_i{}^s$.

7. *A canonical form for the accessory boundary problem.* The accessory

differential equations can be thrown into the Hamiltonian form, and in the case of non-tangency the boundary conditions considerably condensed.

We set

$$\zeta_i = \Omega_{\eta_i'}(\eta, \eta', \mu, \sigma) \qquad \Phi_\beta(\eta, \eta') = 0. \tag{7.1}$$

The relations (7.1) can be solved for η_j' and μ_β in terms of ζ_i and η_i, provided we make the usual assumption that along g

$$\begin{vmatrix} R_{11} & \cdots & R_{1n} & \phi_{1y_1'} & \cdots & \phi_{my_1'} \\ \vdots & & \vdots & \vdots & & \vdots \\ R_{n1} & \cdots & R_{nn} & \phi_{1y_n'} & \cdots & \phi_{my_n'} \\ \phi_{1y_1'} & \cdots & \phi_{1y_n'} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_{my_1'} & \cdots & \phi_{my_n'} & 0 & \cdots & 0 \end{vmatrix} \neq 0. \tag{7.2}$$

With η_i' and μ_β taken as such functions of ζ_i and η_i, we set

$$H(x, \eta, \zeta, \sigma) = \zeta_i \eta_i' - \Omega(\eta, \eta', \mu, \sigma). \tag{7.3}$$

As is well known and easily verified the accessory differential equations take the form *

$$d\eta_i/dx = H_{\zeta_i}; \qquad d\zeta_i/dx = -H_{\eta_i}. \tag{7.4}$$

The boundary conditions (4.5) and (4.6) take the respective forms:

$$\zeta_i^1 c_{ih}^1 - \zeta_i^2 c_{ih}^2 = b_{hk} u_k \qquad (h, k = 1, \cdots, r) \tag{7.5}$$

$$\eta_i^s = c_{ih}^s u_h \qquad (s = 1, 2;\ i = 1, \cdots, n). \tag{7.6}$$

In the case of non-tangency these boundary conditions are equivalent to exactly $2n$ linearly independent conditions $L_p(\eta, \zeta) = 0$ on the variations η_i^s, ζ_i^s $(p = 1, \cdots, 2n)$.

For in the case of non-tangency the matrix $\mathbf{c} = \| c_{ih}^s \|$ is of rank r, so that (7.6) gives $2n - r$ independent conditions on the variables η_i^s. We can replace the variables (u) in (7.5) by linear combinations of the variables η_i^s obtained from (7.6). The resulting relations will be independent of each other because the matrix of the variables ζ_i^s in (7.5) is obtained from $\mathbf{c}$ by interchanging rows and columns and then changing the sign of the last n columns. We thus have $2n$ independent relations as desired.

* See **Bolza**, p. 593, *loc. cit.*

The accessory boundary value problem thus takes the compact fo

$$(7.7) \qquad d\eta_i/dx = H_{\zeta_i}; \qquad d\zeta_i/dx = -H_{\eta_i} \qquad (i = 1, \cdots, n)$$

$$(7.8) \qquad L_p(\eta, \zeta) = 0 \qquad (p = 1, \cdots, 2n).$$

Sufficient Conditions.

8. *The theorem.* By the *Clebsch sufficient condition* we shall hereafter mean the condition that

$$(8.1) \qquad R_{ij}w_iw_j > 0 \qquad (i, j = 1, \cdots, n)$$

for any set $(w) \neq (0)$ satisfying

$$(8.2) \qquad \phi_{\beta y_i'} w_i = 0 \qquad (\beta = 1, \cdots, m)$$

where the partial derivatives involved are evaluated along g.

By the *Weierstrass sufficient condition* we shall mean the condition

$$(8.3) \qquad E(x, y, \bar{y}', y', \lambda) > 0$$

for distinct admissible sets $(x, y, \bar{y}')$ and (x, y, y') for which $(x, y, \bar{y}', \lambda)$ is near the corresponding set on g.

The extremal g will be said to be *identically normal* if every subinterval of it is normal relative to the Euler conditions.

In § 12 we shall prove the following theorem.

Theorem 3. *In order that an identically normal extremal g afford a proper, strong, relative minimum, it is sufficient that the Weierstrass and Clebsch sufficient conditions and the non-tangency condition hold, and that all characteristic roots be positive.*

9. *The fundamental quadratic form.* We are now assuming that g is identically normal and the Clebsch sufficient condition holds. On such an extremal (Morse III, § 4) there exists a positive lower bound of the distances between a point and its nearest conjugate point. With this understood let

$$(9.1) \qquad x = a_0, \cdots, x = a_{p+1} \qquad a_0 = a^1 \qquad a_{p+1} = a^2$$

be a set of successive n-planes cutting the x-axis in the order of their subscripts, and placed so near together that no point and its first following conjugate point lie on a closed segment of g cut out by two successive n-planes.

On the intermediate n-planes

$$(9.2) \qquad x = a_1, \cdots, x = a_p$$

respectively let

$$(9.3) \qquad P_1, \cdots, P_p$$

be a set of points neighboring g. Let (v) be a set of $pn + r = \delta$ variables of which the first r equal $(\alpha_1, \cdots, \alpha_r)$. The next n shall be the respective differences between the y coördinates of P_1 and of the point on g at $x = a_1$, the next n the differences between the y coördinates of P_2 and of the point on g at $x = a_2$, the next n a similar set for P_3 and so on to P_p.

A set (α) in (2.4) determines two end points

$$(9.4) \qquad (x^1, y^1) = P_0 \qquad (x^2, y^2) = P_{p+1}.$$

The complete set (v) determines the points

$$(9.5) \qquad P_0, \cdots, P_{p+1}.$$

If the points (9.5) be sufficiently near g they can be successively joined by extremal segments. Denote the resulting broken extremal by E. We shall say that (v) *determines* the above broken extremal E. The expression J taken along E will be denoted by $J(v)$. With the aid of the Euler equations one sees that the first partial derivatives of $J(v)$ with respect to $v_{r+1}, \cdots, v_\delta$ are all zero for $(v) = (0)$, and with the additional aid of the transversality conditions one sees that the remaining first partial derivatives are zero for $(v) = (0)$. Thus $J(v)$ has a critical point when $(v) = (0)$.

The terms of the second order of $J(v)$ now come to the fore. They will be obtained by means of the following identity in the variables $(z) = (z_1, \cdots, z_\delta)$

$$(9.6) \qquad J^0{}_{v_p v_q} z_p z_q = (d^2/de^2) J(ez_1, \cdots, ez_\delta) \qquad (p, q = 1, \cdots, \delta;\ e = 0)$$

where e is to be set equal to zero after the differentiation, and the partial derivatives on the left are to be evaluated for $(v) = (0)$, as indicated by the superscript 0.

Consider then the family of broken extremals E through the points (9.5) determined by a set $(v) = (ez_1, \cdots, ez_\delta)$ for a fixed (z) and variable e. Represent this family in the form

$$y_i = y_i(x, e) \qquad \alpha_h = ez_h \qquad (h = 1, \cdots, r;\ i = 1, \cdots, n).$$

Although the functions $y_i(x, e)$ fail in general to be of class C' at the corners of the broken extremals, one verifies the fact that (3.8) and (3.9) still hold. The derivation of A must be varied slightly in that we have here

$$(9.7) \qquad \int_{a_1}^{a_2} (F_{y_i} y_{iee} + F_{y_i'} y_{ixee})\, dx = [y_{iee} F_{y_i'}]_1^2 + \sum_{t=1}^{p} [y_{iee} F_{y_i'}]_{a_t+}^{a_t-}$$

while the last sum would not appear if there were no corners.

But even in the case of corners the last sum vanishes. For since the broken extremal is *determined* by (ez) we see, for example, for $x = a_1$, that

$$
(9.8)\qquad
\begin{aligned}
y_1(a_1, e) - y_1(a_1, 0) &= ez_{r+1}\\
&\cdots\cdots\cdots\\
y_n(a_1, e) - y_n(a_1, 0) &= ez_{r+n}
\end{aligned}
$$

so that $y_{iee}(a_1, e) = 0$. Similarly

$$y_{iee}(a_t, e) = 0 \qquad (t = 1, \cdots, p)$$

from which it appears that the last sum in (9.7) vanishes just as in the absence of corners.

Now u_h was defined as $\alpha_h'(0)$. For the present family $u_h = z_h$ so that we can infer from (9.6) that

$$
(9.9)\qquad J^0{}_{v_p v_q} z_p z_q = b_{hk} z_h z_k + 2\int_{a^1}^{a^2} \omega(\eta, \eta')\,dx \qquad
\begin{array}{l}(p, q = 1, \cdots, \delta)\\ (h, k = 1, \cdots, r).\end{array}
$$

A curve $\eta_i = \eta_i(x)$ in the (x, η) space will be called a *secondary extremal* provided it forms the part (η) of a set (η, μ) of functions of x of class C' which satisfy (4.3). From its origin in (9.9) we see that (η) in (9.9) defines a broken secondary extremal E'. To determine its ends and corners we note the following.

Since $u_h = z_h$, (3.12) becomes

$$(9.10)\qquad \eta_i{}^s = c_{ih}{}^s z_h \qquad (i = 1, \cdots, n;\ s = 1, 2;\ h = 1, \cdots, r)$$

so that $(z_1, \cdots, z_r)$ determines the end points of E'. From (9.8) we see that

$$[\eta_1(a_1), \cdots, \eta_n(a_1)] = [z_{r+1}, \cdots, z_{r+n}].$$

Similarly we see that the set $\eta_i(a_2)$ equals the next set of n z's, and so on to the set $\eta_i(a_p)$ which equals the last n z's. Thus the set (z) determines both the end points and the corner points of the broken secondary extremal E'. The set (z) thus uniquely *determines* the broken secondary extremal E'.

We summarize in the following theorem.

THEOREM 4. *If $J(v)$ represents the value of J taken along the broken extremal determined by (v) its terms of second order have the form* (9.9) *where (η) is taken along the broken secondary extremal determined by (z).*

10. *Further introduction of σ.* The previous introduction of the para-

meter σ into the second variation will automatically result if we replace the integrand f by a one-parameter family of integrands

$$f - \sigma[y_i - \bar{y}_i(x)][y_i - \bar{y}_i(x)] \tag{10.1}$$

where $y_i = \bar{y}_i(x)$ as before represents the extremal g. For each value of σ, g will still be an extremal. To construct the broken extremal of the preceding section for this new problem we need the following lemma.

LEMMA. *A decrease of σ never causes a decrease of the distance on the x-axis between two successive conjugate points.*

First recall that we are assuming the Clebsch sufficient condition holds along g, and note that the Clebsch condition is independent of σ. Using Taylor's formula one sees that the Weierstrass sufficient condition is satisfied for the problem of minimizing $I(\eta, u, \sigma)$ subject to (4.1) along any secondary extremal. Moreover, g is assumed identically normal. It follows that any segment d of the x-axis free from successive conjugate points for $\sigma = \sigma_0$ will afford a proper minimum to $I(\eta, 0, \sigma_0)$ in the fixed end point problem. See Morse III.

From the way σ enters into the second variation it follows next that after σ is decreased from σ_0, d will still afford a proper minimum to $I(\eta, 0, \sigma)$.

If, however, there were for the decreased σ a pair of conjugate points on d, it would be possible to make $I(\eta, 0, \sigma)$ zero by taking it along the secondary extremal joining the two conjugate points, and along the x-axis for the rest of d. This is contrary to the nature of a proper minimum. Thus there can be no pairs of conjugate points on d.

The lemma follows readily.

By virtue of this lemma the broken extremal determined by (v) in § 9 for $\sigma = 0$ will also be similarly determined by (v) for each negative σ.

The value of J taken along the broken extremal determined by (v) will now be denoted by $J(v, \sigma)$, $\sigma \leqq 0$. We set

$$Q(z, \sigma) = J_{v_p v_q}(0, \sigma) z_p z_q \qquad (p, q = 1, \cdots, \delta). \tag{10.2}$$

By virtue of Theorem 4 extended to any negative σ we have

$$Q(z, \sigma) = b_{hk} z_h z_k + 2 \int_{a^1}^{a^2} \Omega(\eta, \eta', \mu, \sigma)\, dx \qquad (h, k = 1, \cdots, r) \tag{10.3}$$

where (η) is taken along the broken secondary extremal determined by (z) for the given σ.

11. *Properties of the form $Q(z,\sigma)$.* THEOREM 5. *The quadratic form $Q(z,\sigma)$ is singular if and only if σ is a characteristic root in the accessory boundary value problem.*

The conditions that the form $Q(z,\sigma)$ be singular are that the linear equations

$$Q_{z_p}=0 \qquad (p=1,\cdots,\delta) \tag{11.1}$$

have at least one solution $(z)\neq(0)$.

If such a solution (z) be given we shall first show that the broken secondary extremal E determined by (z), with the set (u) equal to the first r of the z's, gives a characteristic solution.

Let us first examine the geometric meaning of the conditions (11.1) for $p=r+1,\cdots,r+n$. From (11.1) and (10.3) we see that

$$Q_{z_{r+i}}=2[\Omega_{\eta_i'}]_{a_1^+}^{a_1^-}=0 \qquad (i=1,\cdots,n). \tag{11.2}$$

Equations (11.2) taken with (7.2) show that E has no corner at $x=a_1$. More generally we see that the conditions (11.1) with $p>r$ imply the absence of corners at each of the points of E at which

$$x=a_1,\cdots,x=a_p,$$

that is, the complete absence of corners on E.

There remain the conditions (11.1) for which $p\leqq r$. From (10.3) and (11.1) we see that

$$Q_{z_h}=2b_{hk}z_k+2[\Omega_{\eta_i'}\,\partial\eta_i^s/\partial z_h]_1^2=0 \qquad (h,k=1,\cdots,r). \tag{11.3}$$

With the aid of the relations (9.10), namely,

$$\eta_i^s=c_{ih}^s z_h \qquad (h=1,\cdots,r) \tag{11.4}$$

we see that (11.3) takes the form

$$[\Omega_{\eta_i'}c_{ih}^s]_2^1=b_{hk}z_k. \tag{11.5}$$

But (11.4) and (11.5) regarded as boundary conditions on (η) have precisely the form of the boundary conditions (4.5) and (4.6) of the accessory boundary problem.

Finally we see that on E, $(\eta)\neq(0)$ since $(z)\neq(0)$. If then Q is singular we have a characteristic solution (η) of the accessory boundary problem.

Conversely, let there be given for some σ a characteristic solution (η) with its constants (u). Let (z) be a set which determines the secondary extremal (η). In (z) the first r variables will be the set (u).

Conditions (11.5) and (11.4) are satisfied since (η) is a characteristic solution. Conditions (11.3) then follow. All conditions such as (11.2) are satisfied because of the absence of corners on the secondary extremal (η). Hence all conditions (11.1) are satisfied. Moreover, $(z) \neq (0)$ since $(\eta) \neq (0)$.

Thus $Q(z, \sigma)$ is singular when σ is a characteristic root.

The theorem is thereby proved.

The number of linearly independent sets (η) which furnish characteristic solutions corresponding to a given value of σ will be called the index of σ.

For a given $\sigma < 0$ it is clear that linearly independent secondary extremals will determine and be determined by linearly independent sets (z). Since the nullity of the form Q is the number of linearly independent solutions (z) of the equations (11.1) we have the following theorem.

THEOREM 6. *The nullity of the form $Q(z, \sigma)$ equals the index of the root σ.*

Let $M(\eta, \nu)$ be any form quadratic in η_i, ν_i with coefficients continuous in x, and with no terms quadratic in (ν) alone. We shall prove the following lemma involving Ω and $M(\eta, \nu)$.

LEMMA A. *For σ sufficiently large and negative the form*

$$N = 2\Omega(\eta, \nu, \mu, \sigma) + M(\eta, \nu) \tag{11.6}$$

is positive definite in its variables (η, ν, μ), subject to the conditions

$$\phi_{\beta y_i}\eta_i + \phi_{\beta y_i'}\nu_i = 0 \qquad (i = 1, \cdots, n;\ \beta = 1, \cdots, m). \tag{11.7}$$

Subject to (11.7) the form N may be written as follows:

$$N = R_{ij}\nu_i\nu_j + 2Q_{ij}\eta_i\nu_j + P_{ij}\eta_i\eta_j + M(\eta, \nu) - 2\sigma\eta_i\eta_i. \tag{11.8}$$

If we set

$$-\sigma = 1/\rho^2 \qquad \eta_i = \rho\omega_i \qquad \rho \neq 0 \tag{11.9}$$

we have in place of (11.8)

$$N = R_{ij}\nu_i\nu_j + \rho^2 q_{ij}\omega_i\omega_j + \rho p_{ij}\omega_i\nu_j + 2\omega_i\omega_i \tag{11.10}$$

where q_{ij} and p_{ij} are continuous in x. Conditions (11.7) become

$$\rho\phi_{\beta y_i}\omega_i + \phi_{\beta y_i'}\nu_i = 0. \tag{11.11}$$

2

For $\rho = 0$ the form (11.10), taken subject to (11.11), becomes the form

$$N = R_{ij}\nu_i\nu_j + 2\omega_i\omega_i \qquad \phi_{\beta y_i'}\nu_i = 0,$$

and is positive definite by virtue of the Clebsch condition. It follows that for ρ sufficiently small the form (11.10) is positive definite * subject to (11.11), and hence for σ sufficiently large and negative, (11.8) is positive definite subject to (11.7).

Thus the lemma is proved.

We come to a fundamental theorem.

THEOREM 7. *If an extremal g is identically normal, while the Clebsch sufficient condition and the non-tangency condition hold, then the form $Q(z, \sigma)$ will be positive definite for sufficiently large negative values of σ.*

According to Lemma B of § 6, in the case of non-tangency we have

$$Q(z, \sigma) = q(\eta) + 2\int_{a^1}^{a^2} \Omega(\eta, \eta', \mu, \sigma)\,dx \tag{11.12}$$

where $q(\eta)$ is a form quadratic in the variables η_i^s.

Now any such form as q will satisfy a relation

$$q(\eta) \geqq -c[\eta_i^2\eta_i^2 + \eta_i^1\eta_i^1] \qquad (i = 1, \cdots, n) \tag{11.13}$$

provided simply that c be a positive constant sufficiently large.

Let $h(x)$ be any function of x of class C' on the closed interval (a^1, a^2), taking on the values 1 and -1 respectively at $x = a^2$ and $x = a^1$. Then (11.13) may also be written in the form

$$q(\eta) \geqq -c\int_{a^1}^{a^2} (d/dx)\,(\eta_i\eta_i h(x))]\,dx \tag{11.14}$$

where (η) represents any set of functions $\eta_i(x)$ of class D' taking on the end values η_i^s.

From (11..12) and (11.14) we see that

$$Q(z, \sigma) \geqq \int_{a^1}^{a^2} [2\Omega(\eta, \eta', \mu, \sigma) - c(d/dx)\,(\eta_i\eta_i h(x))\,)\,dx \tag{11.15}$$

* This statement can be proven as follows. By virtue of (11.11), for a fixed x, m of the variables (ν) can be eliminated and N reduced to a form L in the remaining variables with coefficients continuous in ρ. The form L is now subject to no auxiliary conditions. For $\rho = 0$ L is positive definite, and according to the ordinary theory of quadratic forms will still be positive definite for ρ sufficiently small.

By virtue of Lemma A of this section the integrand in (11.15) will be positive definite in (η, η', μ), subject to (4.1), provided only that σ be sufficiently large and negative. For such a σ, for $(z) \neq (0)$ and hence $(\eta) \neq (0)$, we have then

$$Q(z, \sigma) > 0.$$

Thus the theorem is proved.

12. *Sufficient condition for a minimum. We shall now prove Theorem* 3.

By virtue of Theorem 5, $Q(z, \sigma)$ is non-singular for $\sigma \leqq 0$, since all characteristic roots are positive. According to Theorem 7, $Q(z, \sigma)$ is positive definite for σ sufficiently large and negative. If we increase σ from large negative values to 0, Q will still remain positive definite since it remains non-singular. Hence under the hypothesis of Theorem 3

$$Q(z, 0) > 0 \qquad\qquad (z) \neq (0).$$

Since the extremal g is identically normal and the Clebsch and Weierstrass conditions hold, a neighborhood N of g exists so small that if the end points and corners of a broken extremal E determined by (z) lie in N, each of E's component segments will afford a minimum to J in N in the fixed end point problem with the given differential conditions.

Let g' now be an admissible curve lying in N. The end points of g' and its intersections with the n-planes

$$x = a_1, \cdots, x = a_p \tag{12.1}$$

of § 9 will determine the set (v) of § 9. For our choice of the neighborhood N we have

$$J_{g'} \geqq J(v). \tag{12.2}$$

On the other hand $Q(v, 0)$ gives the terms of second order in $J(v)$, so that for (v) sufficiently near $(v) = (0)$

$$J(v) \geqq J(0) \tag{12.3}$$

where the equality holds only for $(v) = (0)$.

Thus if g' lies in a sufficiently small neighborhood of g

$$J_{g'} \geqq J(0). \tag{12.4}$$

Now (12.4) becomes an equality only if both (12.2) and (12.3) become equalities. But (12.3) becomes an equality only if $(v) = (0)$, and (12.2)

then becomes an equality only if g' is identical with g. Hence the equality in (12.4) holds only if g' is identical with g.

Thus the minimum is proper and the theorem is proved.

If we recall that the Clebsch sufficient condition entails the Weierstrass sufficient condition in its weak form, that is for sets (x, y, y') neighboring those on g, we have the following corollary.

COROLLARY 1. *For a weak, relative minimum it is sufficient that g be identically normal, that the Clebsch * and non-tangency conditions hold, and that all characteristic roots be positive.*

It is now easy to obtain certain theorems about the sign of the second variation

$$I(\eta, u, 0) = b_{hk}u_h u_k + 2\int_{a^1}^{a^2} \omega(\eta, \eta')\,dx \qquad (h, k = 1, \cdots, r) \tag{12.5}$$

taken subject to the conditions

$$\Phi_\beta = 0 \qquad \eta_i^s = c_{ih}^s u_h \qquad (\beta = 1, \cdots, m;\ i = 1, \cdots, n). \tag{12.6}$$

In the first place we see from (10.3) that along a broken secondary extremal determined by (z) for $\sigma = 0$

$$I(\eta, u, 0) = Q(z, 0). \tag{12.7}$$

In the second place we recall that the Clebsch sufficient condition for g in the original problem entails the Weierstrass sufficient condition

$$E(x, \eta, \bar{\eta}', \eta', \mu) > 0 \qquad (\eta') \neq (\bar{\eta}')$$

set up for the second variation, for x on our interval, for (μ) unrestricted, and for differentially admissible sets (x, η, η') and $(x, \eta, \bar{\eta}')$.

Now let $I(\eta, u, 0)$ be taken along any curve g of class C' which satisfies (12.6). The resulting value of I will certainly be as great as the value of I taken along the broken secondary extremal with the same ends as g and the same intersections with the n-planes (12.1), as a use of the Weierstrass condition shows. But $Q(z, 0)$ is positive definite if all characteristic roots are positive. Hence the following theorem.

THEOREM 7. *For the second variation to be positive for every admissible $(\eta) \not\equiv (0)$ it is sufficient that g be identically normal, that the Clebsch and non-tangency conditions hold, and that all the characteristic roots be positive.*

---

* We refer only to Clebsch and Weierstrass *sufficient* conditions from this point on.

If $\sigma = 0$ is a characteristic root of index q, but there are no negative characteristic roots, we see that the form $Q(z, 0)$ is still positive, except on the q-plane of points (z) satisfying (11.1), on which it is zero. Use of the Weierstrass condition leads to the following theorem.

THEOREM 8. *If g be identically normal, if the Clebsch and non-tangency conditions hold, and if $\sigma = 0$ is the smallest characteristic root, then the second variation will be positive for all admissible sets $(\eta) \not\equiv (0)$, except for those characteristic solutions for which $\sigma = 0$, and for these solutions the second variation will be zero.*

The following corollary is of interest.

COROLLARY. *In order that g afford a proper, strong, relative minimum, it is sufficient that it be identically normal, that the Clebsch, Weierstrass, and non-tangency conditions hold, and that the second variation be positive for all admissible $(\eta) \not\equiv (0)$.*

For if the second variation be positive for all admissible $(\eta) \not\equiv (0)$ there can be no negative root σ by virtue of the proof of Theorem 2, and no root $\sigma = 0$ by virtue of the preceding theorem. Theorem 3 then yields the corollary.

13. *The case $G \equiv 0$. The fixed end point problem.* It will be of interest to determine the nature of the accessory boundary value conditions (7.5) and (7.6) in certain important problems. One can find b_{hk} from (3.10).

In the fixed end point problem one finds that

$$b_{hk} = 0 \qquad c_{ih}{}^{s} = 0,$$

and that accordingly the accessory boundary value conditions reduce simply to $\eta_i{}^{s} = 0$.

The end points variable on two manifolds. We suppose the end points variable on two n-dimensional manifolds not tangent to g. It will be illuminating if we suppose the extremal carried into the x-axis and the tangent planes of the two manifolds into the planes $x = a^{s}$ respectively. Such a transformation can be readily set up, and would be legitimate if the problem came to us in parametric form.

We take the y-coördinates on the respective manifolds as our terminal

parameters (α). We then have $x_h^s(0)=0$. The variations η_i^s are independent, and the accessory boundary conditions are readily seen to have the form

$$\zeta_i^1 = -[F]^1 x_{ij}^1 \eta_j^1$$
$$\zeta_i^2 = -[F]^2 x_{ij}^2 \eta_j^2$$

thus exhibiting the dependence of the accessory boundary conditions upon the curvature of the end manifolds.

The periodic case. Here we suppose that the integrand f and the functions ϕ_β have a period ω in x, and that $a^2 - a^1 = \omega$. We suppose further that an extremal g of period ω is given.

We compare g with neighboring curves of class C' whose end points are congruent, that is, whose y-coördinates at $x = a^2$ and $x = a^1$ are the same. We can take these common y-coördinates as the terminal parameters (α). Thus the terminal conditions may be taken as

$$y_i^s = \alpha_i \qquad x^s = a^s \qquad (i = 1, \cdots, n;\ s = 1, 2).$$

From (3.10) one sees that $b_{hk} = 0$. The accessory boundary conditions become

$$\zeta_i^1 = \zeta_i^2 \qquad \eta_i^1 = \eta_i^2$$

and thus require a characteristic solution to be periodic.

We have thus the following theorems.

THEOREM 9. *In order that a normal periodic extremal afford a minimum to J relative to neighboring differentially admissible curves joining congruent points, it is necessary that the accessory differential equations have no periodic solutions for $\sigma < 0$.*

THEOREM 10. *In order that a periodic extremal afford a proper, strong, minimum to J relative to neighboring differentially admissible curves joining congruent points, it is sufficient that it be identically normal, that it satisfy the Clebsch and Weierstrass sufficient conditions, and that the accessory differential equations possess no periodic solutions for $\sigma \leqq 0$.*

III. THE GENERAL PROBLEM.

14. *The problem defined.* Apart from conditions, such for example as the Clebsch condition, which in the most important problems are generally

fulfilled, the problem of minimizing J is essentially the problem of finding an extremal for which the type * number and nullity of $Q(z, 0)$ are zero.

This is a special case of the more general problem of finding an extremal for which the type number of $Q(z, 0)$ *is a prescribed positive integer, and the nullity zero.*

This problem will have more point if the type number and nullity of $Q(z, 0)$ are really determined by the extremal and boundary conditions alone, and not also by the number and position of the n-planes (9.1) we have used in constructing the broken extremal and defining $J(v, \sigma)$; and this is the case as we shall show. This is subject to the natural limitation that the n-planes (9.1) be sufficiently near together.

The study of this general problem exhibits more adequately the interrelation between the calculus of variations and the accessory boundary problem.

Up to this point the functions $J(v, \sigma)$ and the quadratic form $Q(z, \sigma)$ of § 10 have been defined only for $\sigma \leqq 0$. We can, however, define them also for $\sigma > 0$.

For each $\sigma > 0$ it will, however, first be necessary to choose the n-planes (9.1) nearer together than any two successive conjugate points. If the choice of these n-planes be made, say for $\sigma = \sigma_0$, the same choice will suffice, according to the lemma of § 10, for $\sigma < \sigma_0$. Thus the number δ of variables in the set (z) depends upon the choice of σ, but one choice can be made for all values of $\sigma \leqq \sigma_0$.

It will remove ambiguity if we now denote the form Q by $Q(z, \sigma, \delta)$, and term δ the *dimension* of the form Q.

In Part III we shall assume that the extremal g is identically normal, and that the Clebsch sufficient condition and the non-tangency condition hold.

15. *The theorem about* $Q(z, \sigma, \delta)$. A point $(z) \neq (0)$ at which all the partial derivatives of $Q(z, \sigma, \delta)$ with respect to (z) vanish will be called a *critical set* with characteristic σ and dimension δ. The value of σ will be a characteristic root as we have seen.

We now come to the following lemmas.

LEMMA 1. *The form Q has the property that*

---

* The type number of a quadratic form is the number of negative coefficients appearing in the form after it has been transformed by a real linear non-singular transformation into a sum of squared terms only.

$$Q(z,\sigma',\delta) < Q(z,\sigma'',\delta) \tag{15.1}$$

provided $(z) \neq (0)$ *and* $\sigma' > \sigma''$.

Let (η) represent the broken secondary extremal E determined by (z) when $\sigma = \sigma''$. From (5.1) and (10.3) we have

$$I(\eta, u, \sigma') = Q(z,\sigma'',\delta) + 2(\sigma'' - \sigma') \int_{a^1}^{a^2} \eta_i \eta_i dx \tag{15.2}$$

where (u) gives the first r variables in the set (z). From (15.2) we see that

$$I(\eta, u, \sigma') < Q(z,\sigma'',\delta). \qquad (z) \neq (0). \tag{15.3}$$

But from the minimizing properties of the component arcs of E

$$Q(z,\sigma',\delta) \leqq I(\eta, u, \sigma'). \tag{15.4}$$

The lemma follows from the last two inequalities.

By the *sum of a number of sets* (z) will be meant the set (z) obtained by adding sets (z) as if they were vectors.

LEMMA 2. *The form* $Q(z,\sigma,\delta)$ *is negative if evaluated for a sum* $(z) \neq (0)$ *of a finite number of critical sets with characteristic roots less than* σ.

Without loss of generality we can suppose the critical sets in the sum have distinct characteristic roots.

Let (z) be the sum. Let σ' be the largest of the characteristic roots and (z') the corresponding critical set. Let (z'') be the sum of the remaining critical sets so that $(z) = (z') + (z'')$.

From the preceding lemma we have

$$Q(z,\sigma,\delta) < Q(z,\sigma',\delta) \qquad \sigma' < \sigma \tag{15.5}$$

and this inequality proves the lemma if there is but one critical set in the sum, since the right hand form is then zero.

Now as a matter of algebra of quadratic forms

$$Q(z,\sigma',\delta) = Q(z',\sigma',\delta) + z_p'' Q_{z_p}(z',\sigma',\delta) + Q(z'',\sigma',\delta) \qquad (p = 1, \cdots, \delta). \tag{15.6}$$

But since (z') is a critical set for $\sigma = \sigma'$ this equality reduces to

$$Q(z,\sigma',\delta) = Q(z'',\sigma',\delta). \tag{15.7}$$

If we now adopt the method of mathematical induction and assume the

lemma true for a sum involving one less critical set than the original sum, the right hand form is as a consequence negative. The lemma then follows from (15.5).

LEMMA 3. *The members of any finite ensemble of critical sets* (z) *with distinct characteristics* σ *are linearly independent.*

Suppose there were such a linear dependence. Let (z) be the linear combination which is zero. We can regard (z) as a sum of critical sets with distinct characteristics. Let (z') and (z'') now be defined as in the preceding lemma. Equations (15.6) and (15.7) hold as before. But the left hand member of (15.7) is zero since $(z)=(0)$, and the right hand member is negative by virtue of the preceding lemma.

From this contradiction we infer the truth of the lemma.

For a fixed dimension δ there cannot be more than δ sets (z) which are independent, since there are δ variables in the sets (z). From this fact and the preceding lemma we deduce the following.

The number of characteristic roots less than σ_0 *is at most the minimum dimension number* δ *permissible for* σ_0.

From this lemma we also obtain the following:

LEMMA * 4. *The members of any finite set of characteristic solutions* (η) *with distinct roots* σ *are linearly independent.*

We come now to the fundamental theorem.

THEOREM 11. *The type number of the form* $Q(z, \sigma_0, \delta_0)$ *equals the number* h *of characteristic roots less than* σ_0, *counting each with a multiplicity equal to its index.*

We shall keep $\delta = \delta_0$ throughout the proof.

If σ be sufficiently large and negative we have seen that Q is positive definite. If σ be now increased, the form Q will remain non-singular except when σ passes through a characteristic root σ_1. According to Theorem 6 of § 11, the index q_1 of such a root equals the nullity of the form when $\sigma = \sigma_1$. As σ increases through σ_1, it follows from the theory of quadratic forms that the type number of $Q(z, \sigma, \delta_0)$ changes by at most q_1. Thus the type number of $Q(z, \sigma_0, \delta_0)$ is at most h, the sum of these indices.

---

* This lemma could also be proved directly from the differential equations and boundary conditions.

Corresponding to each characteristic root $\sigma < \sigma_0$ of index q, there are q linearly independent critical sets (z). According to Lemma 2 these sets will make $Q(z, \sigma_0, \delta_0)$ negative, as will any linear combination of them not (0), arising from different characteristic roots $\sigma < \sigma_0$.

But according to Lemma 3, the members of any finite ensemble of critical sets with distinct characteristics will be independent. Thus there are h critical sets with $\sigma < \sigma_0$ which are independent. These h critical sets regarded as points (z) taken with the point $(z) = (0)$ determine an h-plane in the space (z). On this h-plane $Q(z, \sigma_0, \delta_0)$ is negative definite.

It follows * that the type number of Q is at least h. But we have seen that is at most h. Thus the type number is exactly h and the theorem is proved.

16. *Comparison theorems.* We seek to connect characteristic roots with conjugate points.

A point $x = b$ will be said to be a conjugate point of $x = a$ of index q for a given value of σ, if there are just q linearly independent secondary extremals $(\eta) \not\equiv (0)$ which vanish at $x = a$ and $x = b$.

We come to the following theorem.

THEOREM 12. *In the fixed end point problem the form $Q(z, \sigma, \delta)$ is singular if and only if for the given σ, $x = a^2$ is conjugate to $x = a^1$. Moreover the nullity of $Q(z, \sigma, \delta)$ equals the index of $x = a^2$ as a conjugate point of $x = a^1$.*

The type number of $Q(z, \sigma, \delta)$ equals the number of conjugate points of $x = a^1$ *preceding $x = a^2$ for the given σ, counting conjugate points according to their indices.*

The first paragraph of the theorem is a consequence of Theorems 5 and 6 applied to the fixed end point problem, inasmuch as the accessory boundary conditions arising from the fixed end point problem are simply $\eta_i{}^s = 0$. The second paragraph of the theorem is proved by a repetition of the proof of Theorem 2 of Morse I, p. 392, except for obvious changes such as replacing the word "order" by "index."

When the boundary conditions reduce to $\eta_i{}^s = 0$ the accessory boundary problem will be called the *boundary problem with null end points.*

The latter half of Theorem 12 taken with Theorem 11 gives the following Theorem.

* Morse I, p. 390, *loc. cit.*

THEOREM 13. *The number of conjugate points of $x = a^1$ which for $\sigma = \sigma_0$ precede $x = a^2$, equals the number of characteristic roots less than σ_0 in the accessory boundary problem with null end points.*

We shall now prove the following:

There exist arbitrarily many conjugate points of $x = a^1$ preceding $x = a^2$ for σ sufficiently large and positive.

Let (a, b) be any closed interval interior to the given interval. I say that for σ sufficiently large and positive there must be a conjugate point of $x = a^1$ on (a, b).

If this were not so the integral

$$\int_a^b \Omega(\eta, \eta', \mu, \sigma)\, dx \tag{16.1}$$

would be positive for all differentially admissible sets (η) not identically zero on (a, b), of class D', and null at a and b. Moreover there exists at least one set (η) in this class. For example one could take a finite succession of short arcs each of which is a secondary extremal when $\sigma = 0$.

Holding any such (η) fast let σ become positively infinite. The term

$$-2 \int_a^b \sigma \eta_i \eta_i dx$$

included in (16.1) will cause (16.1) to become negatively infinite. This is contrary to a previous assertion.

From this contradiction we infer that for σ sufficiently large and positive there will be at least one conjugate point on (a, b).

Hence the statement in italics is true.

We have compared characteristic roots with conjugate points. We can also compare characteristic roots in one boundary problem with characteristic roots in another such problem, as follows.

THEOREM 14. *The number of characteristic roots less than σ_0 in an accessory boundary problem involving r end parameters (u),* (see § 4), *lies between k and $k + r$ inclusive, where k is the number of characteristic roots less than σ_0 in the boundary problem with null end points.*

This theorem will be proved with the aid of the following lemma on quadratic forms.

LEMMA. *Let $Q_2(z)$ be a quadratic form obtained by setting the first r variables (z) in a quadratic form $Q_1(z)$ equal to zero. If the type number of $Q_2(z)$ is k, the type number of $Q_1(z)$ lies between k and $k+r$ inclusive.*

Let h be the type number of Q_1. It follows from the theory of quadratic forms that there is an h-plane, say π_h, through the origin of the space (z) on which Q_1 is negative definite. If the first r of the variables (z) be set equal to zero, these r conditions together with the linear conditions defining π_h will define a plane π of dimensionality at least $h-r$. On π however, Q_2 will be negative definite. Hence $k \geqq h-r$. See Morse I, Lemma I, p. 390.

The variables (z) which actually appear in Q_2 are those coördinates (z) which are arbitrary in the sub-space $(z_1, \cdots, z_r) = (0)$. Since Q_2 is of type k, there exists in this sub-space a k-plane π_k through the origin on which Q_2 is negative definite. Now in this sub-space $Q_1 \equiv Q_2$. Hence Q_1 is negative definite on π_k. Hence $h \geqq k$.

Thus the lemma is proved.

We come now to the proof of the theorem.

Let $Q_1(z)$ be the form $Q(z, \sigma, \delta)$ with $\sigma = \sigma_0$, set up for the given accessory boundary problem. Let $Q_2(z)$ be the form obtained from $Q_1(z)$ by setting the first r variables in $Q_1(z)$ equal to zero. Using the same intermediate n-planes (9. 2) as were used in setting up $Q_1(z)$, let Q be now set up with $\sigma = \sigma_0$, for the fixed end point problem, and in the resulting form let r be added to the subscript of each variable. There will result the previous form $Q_2(z)$.

Now the type number of $Q_2(z)$ is the number k of characteristic roots less than σ_0 in the boundary problem with null end points. The theorem follows from the lemma and Theorem II.

We have the following corollary.

COROLLARY. *The number of characteristic roots less than σ_0 in any accessory boundary problem involving r end parameters, differs from the corresponding number for any other such problem involving s end parameters by at most the larger of the two numbers r and s.*

We shall prove the following theorem.

THEOREM 15. *The number of characteristic roots on any finite interval of the σ-axis, in any accessory boundary problem involving r end parameters, differs from the corresponding number for the boundary problem with null end points by at most r.*

In fact let h_1 and h_2 be respectively the numbers of characteristic roots less than σ_1 and σ_2 in a given boundary problem. $(\sigma_1 < \sigma_2)$. If k_1 and k_2 denote the corresponding numbers for the boundary problem with null end points we see from Theorem 14 that

$$(16.2) \qquad \begin{aligned} h_1 &= k_1 + m_1 \qquad & 0 \leqq m_1 \leqq r \\ h_2 &= k_2 + m_2 \qquad & 0 \leqq m_2 \leqq r. \end{aligned}$$

Now $h_2 - h_1$ is the number, say m, of characteristic roots of the given problem on the interval

$$(16.3) \qquad \sigma_1 \leqq \sigma < \sigma_2.$$

For this number m we have from (16.2) that

$$m = (k_2 - k_1) + (m_2 - m_1) \mid m_1 - m_2 \mid \leqq r$$

which proves the theorem for the interval (16.3).

Now corresponding to any finite interval whatsoever there exists a closely approximating interval which is of the form (16.3), and which contains the same characteristic roots. Thus the theorem is true in general.

We note the following corollary.

COROLLARY. *The number of characteristic roots on any finite interval of the σ-axis for any accessory boundary problem differs from the corresponding number for any other such problem by at most the sum of the numbers of parameters in the end point conditions of the two problems.*

Except for this corollary the limits of the inequalities given in these theorems can be realized by examples, so that these limits are reduced as much as possible.

It is not so with this corollary. From this corollary it would appear that the number of characteristic roots on a finite interval for one boundary problem might differ from that for another by as much as $2n + 2n = 4n$, whereas the actual limit can be shown to be $2n$, and depending upon the end conditions may be less.

This question is one of a series of questions in the theory of these boundary problems which can be effectively treated by the methods of this paper. Among other things it involves a notion of *the boundary problem common to two problems.*

From the point of view of the theory of boundary value problems the degree of generality of the preceding results is evidenced by the following fact. In the definition of the accessory boundary problem of § 4, one can

prescribe the form $\omega(\eta, \eta')$, the differential conditions $\Phi_\beta = 0$, and the constants $c_{ih}{}^s$ and b_{hk}, subject only to the restriction that $b_{hk} = b_{kh}$. To avoid unnecessary complexity we shall also require that $\| c_{ih}{}^s \|$ be of maximum rank r. The problem of minimizing

$$\tfrac{1}{2} b_{hk} u_h u_k + \int_{a^1}^{a^2} \omega(\eta, \eta')\, dx$$

subject to the conditions

$$\phi_\beta = 0 \qquad \eta_i{}^s = c_{ih}{}^s u_h \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r)$$

will admit the x axis between a^1 and a^2 as an extremal with multipliers $\lambda_0 = 1$, $\lambda_\beta \equiv 0$, and for this extremal the accessory boundary problem will have the prescribed form.

This fact also shows the appropriateness of the form into which the accessory boundary conditions were thrown.

This theory will be further developed from the point of view of the geometry of boundary value problems.

ON THE PROBLEM OF LAGRANGE.

By Lawrence M. Graves.

In recent years there has been a tendency to derive the functional equations which characterize the solutions of problems of the calculus of variations under the least restrictive hypotheses on those solutions. For the simplest problem in the plane in non-parametric form, Whittemore * derived an equation for the solutions in 1901, under the hypothesis that the minimizing function $y(x)$ has a derivative $y'(x)$ which is bounded, and also continuous except on a set of content zero. Tonelli derived this equation † under the weaker hypothesis that the minimizing function $y(x)$ has bounded difference quotients. Similar results for the problem in parametric form were found by Hahn and by Tonelli.‡ Tonelli obtains the Weierstrass condition also, § and for the non-parametric problem under the still weaker hypothesis that $y(x)$ is absolutely continuous.

In the present paper, we obtain a multiplier rule to characterize the minimizing functions $y_i(x)$ in the problem of Lagrange, supposing only that those functions have bounded difference quotients. Under the same hypothesis the analogue of the Weierstrass condition is derived in § 4. Some properties of normal intervals for admissible functions are derived in § 3. The corollary of the theorem in this section enables us to prove the analogue of the Weierstrass condition under hypotheses on the normality of the minimizing functions which are less restrictive than those usually made.¶ § 5 contains an additional remark on the relation between normality and the Weierstrass condition, as well as a remark on an extension of the condition of Mayer.

The methods of proof and notations are largely those used by Bliss,‖ though some differences are necessary. The ordinary theorems on differential and other functional equations are inadequate for obtaining the results of this paper. However, the equations involved are a special case of those treated

* "Lagrange's Equation in the Calculus of Variations, and the Extension of a Theorem of Erdmann", *Annals of Mathematics*, Ser. 2, Vol. 2 (1901), pp. 130-6.

† *Fondamenti di calcolo delle variazioni*, Vol. II, pp. 318, 557.

‡ Hahn, "Ueber die Herleitung der Differentialgleichungen der Variationsrechnung", *Mathematische Annalen*, Vol. 63 (1907), pp. 253-72; Tonelli, *loc. cit.*, pp. 89, 486.

§ Tonelli, *loc. cit.*, pp. 83, 317, 511, 557.

¶ This possibility was first called to my attention by Dr. M. G. Boyce.

‖ "The Problem of Lagrange in the Calculus of Variations," *American Journal of Mathematics*, Vol. 52 (1930), pp. 673-744.

in my paper on *Implicit Functions and Differential Equations in General Analysis.** I have obtained a direct treatment of these equations without such a general background, but it is more complicated. In studying these equations it is convenient to center attention on the derivatives of the functions y_i. Hence we shall denote these derivatives by z_i.

1. *Formulation of the problem.* We shall consider an integral † to be minimized,

$$I[z] = \int_{x_1}^{x_2} f(x, y, z)\, dx,$$

where

$$y_i(x) = y_{i1} + \int_{x_1}^{x} z_i(x)\, dx, \qquad (i = 1, \cdots, n). \tag{1}$$

This relation (1) between $y(x)$ and $z(x)$ is assumed to hold thruout the paper. For simplicity we shall assume that the integrand function f is defined for (x, y) in an $n+1$-dimensional region $\mathfrak{R}$ and for all values of z, and that f is bounded on every bounded domain. We suppose also that for each (y, z) the function f is measurable in x on every measurable subset of its range of definition, and that f is of class $\mathfrak{C}'$ in (y, z) uniformly on every bounded domain of (x, y, z) points. The last statement means that the partial derivatives f_{y_i} and f_{z_i} are bounded and continuous in (y, z) uniformly on every bounded domain of (x, y, z) points. We consider also m functions ϕ_a, $(m < n)$, having the same properties as f. The functions $f(x, y(x), z(x))$, $\phi_a(x, y(x), z(x))$ are measurable whenever the $z_i(x)$ are bounded and measurable and have $(x, y(x))$ interior to $\mathfrak{R}$.‡

We shall call two measurable functions z_1 and z_2 *equivalent* if they differ only on a set of measure zero, and use the sign $z_1 \sim z_2$ in place of $z_1 = z_2$.

Admissible functions $z(x)$ are bounded and measurable, have $(x, y(x))$ interior to the region $\mathfrak{R}$, where $y(x)$ is defined by (1), and satisfy

$$\phi_a(x, y(x), z(x)) \sim 0, \qquad (a = 1, \cdots, m). \tag{2}$$

They also satisfy the condition

H) there exists a positive number μ such that for almost all x the matrix

---

* *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 514-552. See especially Theorem XIII, p. 542.

† All integrals are to be understood in the Lebesgue sense.

‡ See Caratheodory, *Vorlesungen ueber reelle Funktionen*, p. 665; Graves, "Some Theorems Concerning Measurable Functions", *Bulletin of the American Mathematical Society*, Vol. 32 (1926), pp. 529-533.

$\phi_{\alpha z_i}(x, y(x), z(x))$ has a minor determinant whose absolute value is not less than μ.

2. THE MULTIPLIER RULE. *If $z_0(x)$ minimizes $I[z]$ in the class of admissible functions z making $y(x_2) = y_2$, then there exist constants $(l_0, b_1, \cdots, b_n)$ and bounded measurable functions $\lambda_1, \cdots, \lambda_m$, with $(l_0, \lambda_1, \cdots, \lambda_m)$ not all equivalent to zero, such that*

$$F_{z_i} \sim \int_{x_1}^{x} F_{y_i}\, dx + b_i, \qquad (i = 1, \cdots, n), \tag{3}$$

where $F(x, y, z, \lambda, l_0) \equiv l_0 f(x, y, z) + \lambda_\alpha \phi_\alpha(x, y, z)$.

From the hypothesis H) it is plain that $n - m$ additional functions $\phi_r(x, y, z)$ can be adjoined, having the same properties as the functions ϕ_α, so that the determinant

$$|\, \phi_{iz_j}(x, y_0(x), z_0(x)) \,|$$

has absolute value not less than μ on $(x_1 x_2)$. Then we have

$$\phi_i(x, y_0(x), z_0(x)) - w_{i0}(x) = 0, \qquad (i = 1, \cdots, n),$$

where $w_{\alpha 0}(x) \sim 0$, $(\alpha = 1, \cdots, m)$. If we apply theorem XIII of my paper * to the equations

$$\phi_i(x, y_1 + \int_{x_1}^{x} z\, dx, z) = w_i(x), \tag{4}$$

we find that for w near w_0 they have a unique solution $Z[w, x]$ near $z_0(x)$, which is bounded and measurable in x, and of class $\mathfrak{C}'$ in w uniformly.* Here the norm $\|\, w \,\|$ of a set of bounded measurable functions $w_i(x)$ is taken as the upper bound for all i and x of $|\, w_i(x) \,|$. The functional

$$Y[w, x] \equiv y_1 + \int_{x_1}^{x} Z[w, x] dx \tag{5}$$

is continuous in x and of class $\mathfrak{C}'$ in w uniformly, and the differentials $\eta = dY[w, x; \omega]$, $\zeta = dZ[w, x; \omega]$ are the unique solutions of the equations of variation

$$\phi_{iy_j}\eta_j + \phi_{iz_j}\zeta_j = \omega_i, \quad \eta_i = \int_{x_1}^{x} \zeta_i dx.$$

* *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 542.

† See Hildebrandt and Graves, "Implicit Functions and Their Differentials in General Analysis", *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 127-153, for definition of the class C' of functionals. We shall thruout use square brackets to enclose the arguments of functionals. In writing the arguments of the differentials of functionals, we shall revert to a usage customary in the calculus of variations by writing ω for dw, η for dy, ζ for dz.

It is readily verified that the integral $I[z]$ is of class $\mathfrak{C}'$ on the region of the space of bounded measurable functions $z(x)$ where it is defined. Hence the functional

$$J[w] \equiv I[Z[w]] = \int_{x_1}^{x_2} f(x, Y, Z)\, dx$$

is of class $\mathfrak{C}'$ near w_0, and

$$dJ[w;\omega] = \int_{x_1}^{x_2} (f_{y_j} dY_j + f_{z_j} dZ_j)\, dx. \tag{6}$$

If we multiply (6) by a constant l_0 and add

$$\int_{x_1}^{x_2} (\lambda_i \phi_{iy_j} dY_j + \lambda_i \phi_{iz_j} dZ_j - \lambda_i \omega_i)\, dx = 0, \tag{7}$$

where the λ_i are arbitrary bounded measurable functions, we obtain

$$l_0 dJ[w;\omega] = \int_{x_1}^{x_2} (F_{y_j} dY_j + F_{z_j} dZ_j - \lambda_i \omega_i)\, dx, \tag{8}$$

where $F(x, y, z, \lambda, l_0) = l_0 f(x, y, z) + \lambda_i \phi_i(x, y, z)$.

The equations

$$F_{z_i}(x, Y, Z, \lambda, l_0) = \int_{x_2}^{x} F_{y_i}(x, Y, Z, \lambda, l_0)\, dx - c_i \tag{9}$$

are linear in λ, l_0, c, for every w near w_0. Hence, by the same theorem as before * they have a unique solution $\lambda_i = \Lambda_i[w, l_0, c, x]$ bounded and measurable in x. Using (9) to integrate by parts in (8), we find

$$l_0 dJ[w;\omega] + c_i dY_i[w, x_2;\omega] = -\int_{x_1}^{x_2} \Lambda_i[w, l_0, c, x] \omega_i(x)\, dx \tag{10}$$

for every w, l_0, c, ω.

If we now make use of the hypothesis that z_0 minimizes $I[z]$ in the class of admissible functions making $y(x_2) = y_2$, we find that w_0 must minimize $J[w]$ in the class of bounded measurable functions whose first m components are equivalent to zero and which make

$$Y[w, x_2] = y_2. \tag{11}$$

Hence the matrix

$$\left\| \begin{matrix} dJ[w_0;\omega] \\ dY_i[w_0, x_2;\omega] \end{matrix} \right\|$$

has rank less than $n + 1$ when ω ranges over the bounded measurable functions whose first m components are zero, since otherwise, the equations (11)

* Graves, *Implicit Functions and Differential Equations*, Theorem XIII, p. 542.

with
$$J[w] = J[w_0] + u$$
would have solutions for every u near u_0, by the ordinary implicit function theorem. From this we see that there must exist $n+1$ constants l_0 and c_i, not all zero, such that $l_0 dJ[w_0;\omega] + c_i dY_i[w_0, x_2;\omega] = 0$ for the class of functions ω just mentioned. On comparing with equation (10) we find $\Lambda_r[w_0, l_0, c, x] \sim 0$, $(r = m+1, \cdots, n)$. The functions l_0, Λ_a cannot all be equivalent to zero, since then the constants c_i would also be zero, by (9). This proves the multiplier rule.

COROLLARY 1. *If the functions f and ϕ_a are continuous in x, then the functions z_0 and λ may be redefined at the points of a set of measure zero so as to satisfy equations* (1) *and*

$$(2') \qquad \phi_a(x, y(x), z(x)) = 0 \qquad (a = 1, \cdots, m),$$

$$(3') \qquad F_{z_i} = \int_{x_1}^{x} F_{y_i} dx + b_i \qquad (i = 1, \cdots, n),$$

everywhere on (x_1x_2).

The process of redefinition is in terms of limiting values of the functions z_0 and λ. Obviously this process will not introduce any new discontinuities of z_0 or λ.

COROLLARY 2. *If the functions f, ϕ_a, f_{z_i}, and ϕ_{az_i} are of class $\mathfrak{C}'$ in all their arguments, then the set of points where the determinant*

$$R(x) \equiv \begin{vmatrix} F_{z_iz_j} & \phi_{az_i} \\ \phi_{\beta z_j} & 0 \end{vmatrix}$$

is not zero and the minimizing functions z_0 are continuous constitutes an open set O. On O the functions z_0 are of class $\mathfrak{C}'$ and the functions λ are continuous, and equations (3') *may be differentiated with respect to x.*

This corollary is the extension of the Hilbert theorem on the differentiability of minimizing functions and is proved as usual by means of the implicit function theorem.*

3. *Normal intervals for admissible functions.* We shall say that an interval (x_1x_2) is *normal* † for an admissible function $z(x)$ in case the matrix

$$\| dY_i[w, x_2;\omega] \|$$

has rank n when ω ranges over the bounded measurable functions whose first

* See Bliss, *loc. cit.*, p. 684.

† See Bliss, *loc. cit.*, p. 687.

m components are zero. Here w is defined by equations (4), and dY is the differential of the functional (5).

THEOREM. *If (x_1x_2) is a normal interval for an admissible function $z(x)$, and if z has multipliers l_0, λ_a with which it satisfies (3), then $l_0 \neq 0$. If we require $l_0 = 1$, the multipliers λ_a are unique apart from sets of measure zero. Conversely, if (x_1x_2) is not a normal interval for z, then z has multipliers $l_0 = 0$, λ_a with which it satisfies (3).*

For from (3) we obtain (9) with $\lambda_r \sim 0$ $(r = m+1, \cdots, n)$, and then (10). Hence l_0 cannot be zero if the interval is normal. If there were two sets of multipliers with $l_0 = 1$, their difference would be a set of multipliers with $l_0 = 0$, which has just been shown to be impossible. For the converse, we have that there exist constants c_i not all zero such that $c_i dY_i[w, x_2; \omega] = 0$ for every bounded and measurable ω whose first m components are zero. In equations (7) put $\lambda_i(x) = \Lambda_i[w, 0, c, x]$, where Λ is the solution of (9), and we find

$$\int_{x_1}^{x_2} \Lambda_i \omega_i dx = 0,$$

and hence $\Lambda_r \sim 0$, $(r = m+1, \cdots, n)$.

COROLLARY. *If (x_1x_2) is a normal interval for an admissible function z, then every interval containing (x_1x_2) is also normal.*

For, suppose an interval (x_1x_3) contains (x_1x_2) and is not normal. Then the multipliers λ_a of the last part of the theorem for the interval (x_1x_3) are all equivalent to zero on the interval (x_1x_2). From this we see that the constants c_i of equations (9) must all be zero, and then from the uniqueness of the solution of these equations we find $\lambda_a \sim 0$ on the larger interval (x_1x_3).

4. *The analogue of the Weierstrass condition.* In this section we assume that the functions f and ϕ_a are continuous in x. We suppose also that z_0 minimizes $I[z]$ in the class of admissible functions z giving y the end-values y_1 and y_2, and that z_0 has multipliers $l_0 = 1$, $\lambda_a(x)$, with which it satisfies equations (2′) and (3′) of page 551. *Then if the sub-interval (x_1x_3) of (x_1x_2) is normal for z_0, the functions z_0 and λ can be modified at the points of a set of measure zero without disturbing the validity of equations (2′) and (3′), so that the Weierstrass function*

$$E(x, y_0(x), z_0(x), \lambda(x), \bar{z}) \geqq 0$$

everywhere on the interval (x_3x_2), for every set of numbers $\bar{z}$ such that $\phi_a(x, y_0(x), \bar{z}) = 0$ while the matrix $\phi_{az_i}(x, y_0(x), \bar{z})$ has rank m.

To prove this, consider the point set S where each $y_{i0}(x)$ has a derivative equal to $z_{i0}(x)$ and

$$\int_{x_1}^{x} f(x, y_0(x), z_0(x))\, dx$$

has a derivative equal to $f(x, y_0(x), z_0(x))$, and let x_4 be a point of S between x_3 and x_2, $y_{i4} = y_{i0}(x_4)$, $z_{i4} = z_{i0}(x_4)$, $\lambda_{a4} = \lambda_a(x_4)$. If $\bar{z}$ satisfies the conditions specified in the theorem, the equations

$$\phi_a(x, \ddot{y}, \ddot{z}) = 0, \quad \ddot{y}_i = y_{i4} + \int_{x_4}^{x} \ddot{z}_i dx,$$

have a continuous solution $\ddot{z}(x)$ near x_4, with $\ddot{z}(x_4) = \bar{z}$. Since the interval $(x_1 x_4)$ is normal by the corollary in § 3, the matrix

$$\| \; dY_i[w_0, x_4; \omega] \; \|$$

has rank n. Let $\omega_i^{(1)}, \cdots, \omega_i^{(n)}$, give it this rank. Then the equations $Y_i[w_0 + a_k\omega^{(k)}, x_5] = \ddot{y}_i(x_5)$ have a unique solution $a_k = a_k(x_5)$ for x_5 near x_4 and a_k near zero, and this solution is continuous. By a direct consideration of the difference quotients the derivatives $a_k'(x_4)$ can readily be shown to exist and to satisfy the equations $dY_i[w_0, x_4; a_k'\omega^{(k)}] = \bar{z}_i - z_{i4}$, since $\ddot{z}(x)$ is continuous and x_4 is a point of the set S.

Now set

$$\begin{aligned}
\mathfrak{Z}(x, x_5) &= Z[w_0 + a_k(x_5)\omega^{(k)}, x], \quad \mathfrak{Y}(x, x_5) = Y[w_0 + a_k(x_5)\omega^{(k)}, x], \\
z(x, x_5) &= \mathfrak{Z}(x, x_5) \quad \text{on } x_1 \leqq x \leqq x_5, \\
&= \ddot{z}(x) \qquad\quad \text{on } x_5 < x \leqq x_4, \\
&= z_0(x) \qquad\quad \text{on } x_4 < x \leqq x_2.
\end{aligned}$$

Then the function $z(x, x_5)$ is admissible for $x_5 \leqq x_4$, and satisfies the end conditions

$$\int_{x_1}^{x_2} z_i(x, x_5)\, dx = y_{i2} - y_{i1}.$$

Hence $K(x_5) \equiv I[z(x, x_5)] \geqq I[z_0(x)] = K(x_4)$. By a direct consideration of the difference quotient and application of the theorem of mean value, we find that K has a left-hand derivative at x_4, given by

$$(12) \qquad K'(x_4) = -f(x_4, y_4, \bar{z}) + f(x_4, y_4, z_4) + \int_{x_1}^{x_4} (f_{y_i} dY_i + f_{z_i} dZ_i)\, dx,$$

where the arguments of f_{y_i} and f_{z_i} are $x, y_0(x), z_0(x)$, and those of dY_i and dZ_i are $w_0, x, a_k'\omega^{(k)}$. Now dY and dZ satisfy

$$(13) \qquad \phi_{ay_i} dY_i + \phi_{az_i} dZ_i = 0,$$

where the arguments are respectively the same as in (12). If we multiply

(13) by the multipliers λ_a belonging to z_0, integrate from x_1 to x_4, add to equation (12), and use equations (3′), we find

$$K'(x_4) = -f(x_4, y_4, \bar{z}) + f(x_4, y_4, z_4) + (\bar{z}_i - z_{i4}) F_{z_i}(x_4, y_4, z_4, \lambda_4).$$

Now since

$$\phi_a(x_4, y_4, z_4) = 0, \quad \phi_a(x_4, y_4, \bar{z}) = 0,$$

the derivative

$$K'(x_4) = -E(x_4, y_4, z_4, \lambda_4, \bar{z}).$$

To show $E \geqq 0$ for limiting values of the functions z_0, λ, consider an infinite sequence $\{x_q\}$ of points of the set S, such that $\lim x_q = x^*$, $\lim z_0(x_q) = z^*$, $\lim \lambda(x_q) = \lambda^*$. If $\bar{z}$ satisfies the conditions of the theorem at the point x^*, then there is a sequence $\{\bar{z}_q\}$ such that $\bar{z}_q$ satisfies the conditions of the theorem at the point x_q, and $\lim \bar{z}_q = \bar{z}$. Hence $E(x_q, y_0(x_q), z_0(x_q), \lambda(x_q), \bar{z}_q) \geqq 0$, and since E is continuous, $E(x^*, y_0(x^*), z^*, \lambda^*, \bar{z}) \geqq 0$.

5. *Remarks.* An example in which the preceding proof leads to the condition of Weierstrass when the proofs usually given do not, may be obtained by considering the isoperimetric problem. Let $z_0(x)$ have a single discontinuity at x_3, and minimize the integral I in the class of curves giving a second integral

$$G[z] = \int_{x_1}^{x_2} g(x, y, z)\, dx$$

a prescribed value. In case $z_0(x)$ is a mimimizing function for G on every interval not containing x_3, then every such interval is abnormal. But if $g_z(x, y_0(x), z_0(x))$ has a non-removable discontinuity at x_3, every interval containing x_3 is normal, and the Weierstrass condition holds on $(x_1 x_2)$. Cases of this sort would be those considered by Caratheodory in the second part of his dissertation,* in which the extremals for the two integrals I and G are the same.

I have also considered the second variation and obtained an extension of the necessary condition of Mayer. However, this extension is not satisfying, and the case when the derivatives have only a finite number of ordinary discontinuities † shows that a more penetrating study must be made before a consistent set of sufficient conditions can be given for the case when there are infinitely many discontinuities.

THE UNIVERSITY OF CHICAGO.

* *Ueber die diskontinuierlichen Lösungen in der Variationsrechnung*, Dissertation, Göttingen, 1904.

† See Graves, "Discontinuous Solutions in Space Problems of the Calculus of Variations", *American Journal of Mathematics*, Vol. 52 (1930), pp. 1-28.

PERSPECTIVE ELLIPTIC CURVES.

By Elizabeth Morgan Cooper.

1. *Introduction.* Two curves in a plane are said to be perspective if the points of one and the tangents of the other can be put into one-to-one correspondence in such a way that each point of the one lies on the corresponding tangent of the other. The property is not confined to curves in the plane, (for a discussion of many possible cases of perspective space curves and surfaces, see Segre [13, 14]), but this paper will deal with perspective plane curves only.

A curve perspective to a curve C is a birational transform of C, for if an envelope E is perspective to C a line τ of E, incident with the point t of C, will have, in general, a unique point of contact t' with E. Therefore the points of contact of E will be in one-to-one correspondence with the points of C, and their locus C', the dual form of E, will be a birational transform of C. Hence two perspective curves have, necessarily, the same genus. If, on the other hand, C' is a birational transform of C, the joins of corresponding points of C and C' envelope a curve E perspective to both C and C'. Two birational transforms, C' and C'', of C will, however, lead to the same envelope perspective to C if it happens that corresponding points of C, C' and C'' are collinear.*

Some, at least, of the curves perspective to C are obtained as transforms of C by quadratic null systems, i. e., quadratic correlations in which corresponding point and line are incident.

A conic generated by two perspective points, a cubic generated by a line and conic, a quartic by two conics, or, if it is "singular," by a line and cubic (Schroters [11]) are simple examples of a curve generated by two of its perspective curves. The fact that a curve can be so generated leads to a classification of rational curves by means of their perspective curves. Cf. Haase,[5] Meyer,[9] Stahl [15]). Take, for example, the rational sextic. A rational curve has ∞^{2m-n+1} perspective m-ics (Brill,[1] Coble [4]), hence the general rational sextic has ∞^1 perspective cubics and is the locus of points of intersection of corresponding lines of any two of these cubics. A condition on a sextic

* Haase [5] proves that, given 3 rational curves of orders m, n and p respectively, each perspective to the other two, if it happens $m+n+p+1$ times that corresponding points are collinear, such points are always collinear.

gives it a perspective conic. It can then be generated by that conic and by one of its ∞^3 perspective quartics, while a sextic with a five-fold point can be generated by that point and a perspective quintic. There are, then, three types of rational sextics, the classification depending on the curve of lowest class perspective to the sextic. From this point of view perspective curves are important, and it is desirable to know the distribution of curves of given class (or order) perspective to a curve of given order (or class), and given genus. In this paper the number and arrangement of curves perspective to a curve of genus 1 is found. (p. 562).

It has been proved by Meyer [7, 8] that if two rational class curves, of lowest class,

$$\rho\xi_i = (\alpha_i\tau)^{m_1} \quad \text{and} \quad \rho\xi_i = (\beta_i\tau)^{m_2}, \qquad (i = 0, 1, 2),$$

are perspective to a curve C given by

$$\mu x_i = (a_i t)^n, \qquad (i = 0, 1, 2),$$

i. e., if

$$(a_0t)^n(\alpha_0\tau)^{m_1} + (a_1t)^n(\alpha_1\tau)^{m_1} + (a_2t)^n(\alpha_2\tau)^{m_1} \equiv 0 \quad \text{when} \quad \tau = t$$

and

$$(a_0t)^n(\beta_0\tau)^{m_2} + (a_1t)^n(\beta_1\tau)^{m_2} + (a_2t)^n(\beta_2\tau)^{m_2} \equiv 0 \quad \text{when} \quad \tau = t$$

then all curves of class m $\left(m > \begin{matrix} m_1 \\ m_2 \end{matrix}\right)$ perspective to C are represented by

$$\rho\xi_i = (\gamma_i\tau)^{m-m_1}(\alpha_i\tau)^{m_1} + (\delta_i\tau)^{m-m_2}(\beta_i\tau)^{m_2}, \quad (i = 0, 1, 2),$$

where $(\gamma_i\tau)^{m-m_1}$ and $(\delta_i\tau)^{m-m_2}$ are arbitrary binary forms of the order indicated. For elliptic curves it is possible to write down perspective curves of a given order m in terms of two perspective curves of lower orders m_1 and m_2 in a similar manner, if and only if $m > \begin{matrix} m_1 + 1 \\ m_2 + 1 \end{matrix}$. This restriction is needed, of course, because there is no first order elliptic function to take the place of the linear binary form.

A great deal of work on rational perspective curves has already been done, particularly by Stahl [15, 16]. Coble [4] has put in simpler and more interesting form the work of Stahl by showing its connection with apolarity, a property which does not, unfortunately, carry over to the elliptic case. Coble [4] discusses also the case of a curve which is doubly perspective to a given curve, i. e., a curve each of whose lines cuts the given curve twice in the point corresponding to the line. Such a curve is tangent to the given

curve at every point, and is, in fact, the line form of the curve itself. The condition that a curve have one or more cusps is merely the condition that it have a doubly perspective curve of sufficiently low class and this condition for rational curves Coble [11] writes down in very simple form.

It is proved by Haase [5] and Coble [4] that a rational n-ic and one of its perspective m-ics have $m+n-2$ contacts. Coble [4] finds that they lie in an involution $I_{2m-n+1,2n-3-m}$. The analogous situation for elliptic curves is discussed hereafter. (p. 563).

Some special applications of perspective rational curves are given by Coble,[2, 3] St. Jolles,[6] Schumacher,[12] Study.[17] With the exception of the work of Segre,[13, 14] who gives no specific results which are applicable in the plane, the articles we have located deal with the binary case only.

The elliptic case is more difficult to handle than the rational case because elliptic functions do not lend themselves as conveniently to the working out of the perspective theory. Considerable difficulty arises from the fact that the parametric expression of the coördinates of a curve C_n, given by

$$\mu x_i = R_i[p(u),\ p'(u)], \qquad (i=0,1,2),$$

where R_i is a rational function, or by

$$\mu x_i = a_{i0} + a_{i2}p(u) + a_{i3}p'(u) + \cdots + a_{in}p^{n-2}(u), \qquad (i=0,1,2),$$

suffers a serious alteration when the parameter change

$$u' = \pm u + k,$$

corresponding to the general transformation of the curve into itself, is introduced.

When, in the binary case, we ask that $\rho\xi_i = (\alpha_i\tau)^m$ be perspective to $\mu x_i = (a_i t)^n$, assumed given, the determination of the coefficients of $(\alpha_i\tau)^m$ *includes* the proper parameter choice, for $\rho\xi_i = (\alpha_i\tau)^m$ merely becomes $\rho'\xi_i = (\alpha_i'\tau')^m$ when the transformation

$$\tau = (a\tau' + b)/(c\tau' + d)$$

is made. In the elliptic case, on the other hand, though an appropriate choice of constants in $\rho\xi_i = R_i'[p(v),\ p'(v)]$ will make the curve it represents perspective to $\mu x_i = R_i[p(u),\ p'(u)]$ in the sense that the line v of the one and the point u of the other are incident, we can not, by merely changing *coefficients* and keeping the same *form* of the coördinate expressions, use $u' = u + k$ to replace u as parameter. Indeed when we substitute $u' - k$ for u and use the addition formulae to express μx_i in terms of $p(u')$ and

$p'(u')$, we get expressions which are rational but no longer integral. By multiplying through by a common denominator we obtain, for each coördinate, a rational integral function of $p(u')$ and $p'(u')$, but these functions have higher orders than n, and we can not isolate the extraneous factors which have been introduced and which complicate matters greatly. The use of sigma functions does not seem to be profitable as an alternative, as it gives a still more troublesome form for the incidence relation. The most useful way of writing the coördinates seems to be in terms of the p-function and its derivatives.

It turns out that the elliptic parameter k appears as an independent parameter in the family of m-ics perspective to a given n-ic and leads to a distribution of these m-ics in a manner significantly different from the arrangement for rational curves.

2. *The m-ics Perspective to a Given n-ic.* The general elliptic class m-ic E_m, given by

$$(1)\qquad \rho\xi_i = \alpha_{i0} + \alpha_{i2}p(v) + \alpha_{i3}p'(v) + \cdots + \alpha_{im}p^{m-2}(v), \qquad (i = 0, 1, 2),$$

will be perspective to a given order n-ic C_n, given by

$$(2)\qquad \begin{aligned} \mu x_i = a_{i0} + a_{i2}p(-u+k) + a_{i3}p'(-u+k) \\ + \cdots + a_{in}p^{n-2}(-u+k), \end{aligned} \qquad (i = 0, 1, 2),$$

if and only if the incidence condition

$$(3)\qquad (x\xi)_{v=u} = 0$$

vanishes identically. If, instead of $(-u+k)$, we take $(u+k)$ for canonical parameter on C_n, we get the same curves E_m with canonical parameter $-v$ instead of v.

The incidence relation (3) is an elliptic function with an m-th order pole at $u = 0$ and an n-th order pole at $u = k$. The vanishing of the constant terms and of the coefficients of the principal parts of both expansions is a necessary and sufficient condition of perspectivity and gives $m + n + 2$ equations which are linear and homogeneous in α_{ij} and elliptic in k. Of these equations only $m + n - 1$ are needed to put on (3) the condition that it be a constant. We can, for example, require the vanishing of the m coefficients of $1/u^j$ $(j = 1, 2, 3, \cdots, m)$ in the expansion of (3) about $u = 0$ and of the $n - 1$ coefficients of $1/(u-k)^i$ $(i = 2, 3, \cdots, n)$ in the expansion about $u = k$. One further condition will require that that constant

be zero, hence it is clear that not more than $m+n$ of the $m+n+2$ equations are independent, and we shall show that the rank of the matrix formed from the equations is always $m+n$.

To investigate the rank of this matrix, which we shall call M_{mn}, we use not the general n-ic but the special case

$$\mu x_0 = 1, \quad \mu x_1 = p^{h-2}(-u+k), \quad \mu x_2 = p^{n-2}(-u+k),$$

since for special values of a_{ij} the rank will be less than or equal to the rank in the general case. Since the rank, we find, can never be greater than $2m+h$, we take $h=m$ when $m \leqq n$ and when n is being increased, otherwise, for simplicity, we choose $h=2$. In forming the matrix we take the equations in this order:

$$\begin{aligned}
&A_0 = 0, \\
&(-1)^j A_j/(j-1)\,! = 0, && (j = 2, 3, 4, \cdots, m), \\
&B_j/(j-1)\,! = 0, && (j = h, h-1, h-2, \cdots, 3, 2), \\
&B_1 = 0, \\
&B_j/(j-1)\,! = 0, && (j = n, n-1, n-2, \cdots, h+2, h+1),
\end{aligned}$$

where A_j, $(j = 0, 1, 2, \cdots, m)$, is the coefficient of $1/u^j$ and B_j, $(j = 0, 1, 2, \cdots, n)$, is the coefficient of $1/(u-k)^j$ in the expansions of $(x\xi)_{v=u}$ about $u=0$ and $u=k$ respectively.

The matrix $M_{m,n+1}$ is formed from the matrix M_{mn} by the addition of one row and by changing, in numerical coefficient and by differentiation with respect to k, certain of the elements of the original matrix. The matrices M_{44} and M_{36} will serve to indicate the general situation in the two cases when $h=2$ and $h=m$ respectively. The heavy figures, besides indicating a new row, show wherein the corresponding elements of M_{43} and M_{35} have been altered in a numerical coefficient or in the order of a derivative.

$$= \left\| \begin{array}{cccccccccccc}
1 & 0 & 0 & 2\,!c_2 & p & p''/2 & p'''/3 & (2\,!c_2+p^{\mathrm{IV}}/4) & \boldsymbol{p''} & \boldsymbol{p^{\mathrm{IV}}/2} & \boldsymbol{p^{\mathrm{V}}/3} & (2\,!c_2+\boldsymbol{p^{\mathrm{VI}}}/4) \\
0 & 1 & 0 & 0 & 0 & p & \binom{2}{1}p' & \binom{3}{1}p'' & 0 & \boldsymbol{p''} & \binom{2}{1}\boldsymbol{p'''} & \binom{3}{1}\boldsymbol{p^{\mathrm{IV}}} \\
0 & 0 & 1 & 0 & 0 & 0 & p & \binom{3}{2}p & 0 & 0 & \boldsymbol{p''} & \binom{3}{2}\boldsymbol{p'''} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & p & 0 & 0 & 0 & \boldsymbol{p''} \\
0 & 0 & 0 & 0 & 1 & p & p' & p'' & 0 & \binom{3}{2}\boldsymbol{p''} & \binom{3}{2}\boldsymbol{p'''} & \binom{3}{2}\boldsymbol{p^{\mathrm{IV}}} \\
0 & 0 & 0 & 0 & 0 & p' & p'' & p''' & 0 & \binom{3}{3}\boldsymbol{p'''} & \binom{3}{3}\boldsymbol{p^{\mathrm{IV}}} & \binom{3}{3}\boldsymbol{p^{\mathrm{V}}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \binom{3}{0}p & \binom{3}{0}p' & \binom{3}{0}p'' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \binom{3}{1}\boldsymbol{p'} & \binom{3}{1}\boldsymbol{p''} & \binom{3}{1}\boldsymbol{p'''}
\end{array} \right\|,$$

$$M_{36} = \begin{Vmatrix} 1 & 0 & 0 & p' & p'''/2 & p^{\mathrm{IV}}/3 & p^{\mathrm{IV}} & p^{\mathrm{VI}}/2 & p^{\mathrm{VII}}/3 \\ 0 & 1 & 0 & 0 & p' & \binom{2}{1}p'' & 0 & p^{\mathrm{IV}} & \binom{2}{1}p^{\mathrm{V}} \\ 0 & 0 & 1 & 0 & 0 & p' & 0 & 0 & p^{\mathrm{IV}} \\ 0 & 0 & 0 & 1 & \binom{2}{0}p & \binom{2}{0}p' & 0 & \binom{5}{3}p''' & \binom{5}{3}p^{\mathrm{IV}} \\ 0 & 0 & 0 & 0 & \binom{2}{1}p' & \binom{2}{1}p'' & 0 & \binom{5}{4}p^{\mathrm{IV}} & \binom{5}{4}p^{\mathrm{V}} \\ 0 & 0 & 0 & 0 & \binom{2}{2}p'' & \binom{2}{2}p''' & 0 & \binom{5}{5}p^{\mathrm{V}} & \binom{5}{5}p^{\mathrm{VI}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \binom{5}{0}p & \binom{5}{0}p' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{5}{1}p' & \binom{5}{1}p'' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{5}{2}p'' & \binom{5}{2}p''' \end{Vmatrix}.$$

Here, as elsewhere in this paper, $p, p', \cdots p^n$ refer to $p(k), p'(k), \cdots p^n(k)$ when no other argument is indicated. The element standing in the $(m+1)$-th row and $(2m+2)$-th column is $\binom{n-1}{n-h} p^{n-h}$. The constant c_2 comes from the expansion

$$p(k) = 1/k^2 + c_2k^2 + c_3k^4 + \cdots.$$

The lower right hand block of such a matrix has the form

$$(4) \qquad \begin{Vmatrix} p' & p'' & p''' & \cdots & p^{m-1} \\ p'' & p''' & p^{\mathrm{IV}} & \cdots & p^m \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ p^{n-h-1} & p^{n-h} & p^{n-h+1} & \cdots & p^{n-h+m-3} \end{Vmatrix}$$

after the i-th row has been divided by $\binom{n-1}{i}$, and the block composed of the elements just below the $(m+1)$-th row and to the right of the $(m+1)$-th column has the form

$$(5) \qquad \begin{Vmatrix} p' & p'' & p''' & \cdots & p^{m-1} \\ p'' & p''' & p^{\mathrm{IV}} & \cdots & p^m \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ p^{h-1} & p^{h} & p^{h+1} & \cdots & p^{h+m-3} \end{Vmatrix}$$

when the i-th row has been divided by $\binom{h-1}{i}$. To investigate the rank of such a matrix we substitute for each element the first term of its expansion about $k = 0$ and get

$$\left\|\begin{array}{ccccc} \dfrac{-2!}{u^3} & \dfrac{3!}{u^4} & \dfrac{-4!}{u^5} & \cdots & \dfrac{(-1)^{m-1}m!}{u^{m+1}} \\ \dfrac{3!}{u^4} & \dfrac{-4!}{u^5} & \dfrac{5!}{u^6} & \cdots & \dfrac{(-1)^{m}(m+1)!}{u^{m+2}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{(-1)^{s-1}s!}{u^{s+1}} & \dfrac{(-1)^{s}(s+1)!}{u^{s+2}} & \dfrac{(-1)^{s+1}(s+2)!}{u^{s+3}} & \cdots & \dfrac{(-1)^{m+s-2}(m+s-2)!}{u^{m+s+1}} \end{array}\right\|,$$

where $s=n-h$ in the matrix (4) and $s=h$ in (5). A simple manipulation gives

$$\left\|\begin{array}{ccccccccc} 1/u^3 & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 1/u^5 & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 1/u^7 & \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 & 1/u^{s+\epsilon} & & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 & 0 & 1/u^{s+2+\epsilon} & \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\|.$$

where $\epsilon=1$, s even, $\epsilon=0$, s odd.

Therefore the matrices (4) and (5) can be put into such form as to have zero elements below and to the left of the diagonal line, and diagonal elements which are non-vanishing in general. This means that the last row of the matrix $M_{m,n+1}$ (i. e., the row added to M_{mn}) can be so manipulated as to have zero elements to the left of the diagonal line, and, as diagonal element, a function which is non-vanishing for an arbitrary choice of k. Hence the rank of $M_{m,n+1}$ is one more than the rank of M_{mn}.

If m, instead of n, is increased by 1, the new matrix has one new row, the m-th, and three new columns, the m-th, $2m$-th, and $3m$-th. We write down the matrix M_{65}. Deleting the elements in heavy type gives the matrix M_{55}.

$$M_{65}=\left\|\begin{array}{cccccccccccc} 1 & 0 & 0 & 2!\,c_2 & 0 & \mathbf{4!\,c_3} & p & p''/2 & p'''/3 & (2!\,c_2+p^{IV}/4) & p^{V}/5 & \mathbf{4!\,c_3+p''/6} \\ 0 & 1 & 0 & 0 & 0 & \mathbf{0} & 0 & p & \binom{2}{1}p' & \binom{3}{1}p'' & \binom{4}{1}p''' & \binom{5}{1}\mathbf{p^{IV}} \\ 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & p & \binom{3}{2}p' & \binom{4}{2}p'' & \binom{5}{2}\mathbf{p'''} \\ 0 & 0 & 0 & 1 & 0 & \mathbf{0} & 0 & 0 & 0 & p & \binom{4}{3}p' & \binom{5}{3}\mathbf{p''} \\ 0 & 0 & 0 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 & p & \binom{5}{4}\mathbf{p'} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{p} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 1 & p & p' & p'' & p''' & \mathbf{p^{IV}} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & p' & p'' & p''' & p^{IV} & \mathbf{p^{V}} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{array}\right\|$$

$$\left.\begin{array}{cccccc}
p''' & p^{\mathrm{V}}/2 & p^{\mathrm{VI}}/3 & (2!\,c_2+p^{\mathrm{VII}}/4) & p^{\mathrm{VIII}}/5 & 4!\,c_3+p^{\mathrm{IX}}/6 \\
0 & p''' & \binom{2}{1}p^{\mathrm{IV}} & \binom{3}{1}p^{\mathrm{V}} & \binom{4}{1}p^{\mathrm{VI}} & \binom{5}{1}p^{\mathrm{VII}} \\
0 & 0 & p''' & \binom{3}{2}p^{\mathrm{IV}} & \binom{4}{2}p^{\mathrm{V}} & \binom{5}{2}p^{\mathrm{VI}} \\
0 & 0 & 0 & p''' & \binom{4}{3}p^{\mathrm{IV}} & \binom{5}{3}p^{\mathrm{V}} \\
0 & 0 & 0 & 0 & p''' & \binom{5}{4}p^{\mathrm{IV}} \\
0 & 0 & 0 & 0 & 0 & p''' \\
0 & \binom{4}{3}p''' & \binom{4}{3}p^{\mathrm{IV}} & \binom{4}{3}p^{\mathrm{V}} & \binom{4}{3}p^{\mathrm{VI}} & \binom{4}{3}p^{\mathrm{VII}} \\
0 & \binom{4}{4}p^{\mathrm{IV}} & \binom{4}{4}p^{\mathrm{V}} & \binom{4}{4}p^{\mathrm{VI}} & \binom{4}{4}p^{\mathrm{VII}} & \binom{4}{4}p^{\mathrm{VIII}} \\
1 & \binom{4}{0}p & \binom{4}{0}p' & \binom{4}{0}p'' & \binom{4}{0}p''' & \binom{4}{0}p^{\mathrm{IV}} \\
0 & \binom{4}{1}p' & \binom{4}{1}p'' & \binom{4}{1}p''' & \binom{4}{1}p^{\mathrm{IV}} & \binom{4}{1}p^{\mathrm{V}} \\
0 & \binom{4}{2}p'' & \binom{4}{2}p''' & \binom{4}{2}p^{\mathrm{IV}} & \binom{4}{2}p^{\mathrm{V}} & \binom{4}{2}p^{\mathrm{VI}}
\end{array}\right\|$$

The new last column always falls entirely to the right of the diagonal line and so can not affect the rank; the new $2m$-th column is equally ineffective, while the new m-th row and m-th column supply a 1 as diagonal element and nothing but zero elements to the left of it and below it. Hence the rank of $M_{m+1,n}$ is one more than the rank of M_{mn}. It has been verified (p. 566) that the rank of M_{33} is $3+3=6$. Therefore, for arbitrary k, the rank of M_{mn} is $m+n$. We have, then, $m+n$ linear relations connecting the $3m$ homogeneous constants α_{ij} and, for an arbitrary value of k, ∞^{2m-n-1} m-ics perspective to C_n. That k is an independent parameter can be shown as follows.

Suppose that k and the $2m-n-1$ essential parameters of the family of m-ics corresponding to k are all functions of δ_i $(i=1, 2, 3, \cdots, 2m-n-1)$ and these only. Then $k=k(\delta_i)=0$ would imply a relation among the δ_i's and we should have, not ∞^{2m-n-1} m-ics corresponding to $k=0$, but only ∞^{2m-n-2}. We have, therefore, proved the following theorem:

THEOREM I. *There are ∞^{2m-n} m-ics perspective to a given n-ic. Corresponding to a given transformation of the curve into itself there are ∞^{2m-n-1} perspective m-ics which lie in a linear system.*

The division of the set of perspective m-ics into families corresponding to a given value of k, corresponding, that is, to a given birational transformation of the curve into itself is a peculiarity of the elliptic case. It would not occur in the case of hyperelliptic curves which are transformed into themselves by a finite number of birational transformations * and it does not occur in the rational case.

* *Pascal Repertorium,* Chapter XV, § 6.

3. *The contacts of two Perspective Curves.*

THEOREM II. *An elliptic n-ic and an m-ic perspective to it have, in general, $m + n$ contacts.*

The contacts of C_n, (2), and E_m (1), will be the zeros of a function obtained by differentiating the incidence relation $(x\xi)$ with respect to u before putting $v = u$. The coefficient of $1/u^m$, the first term of the principal part of the expansion of this function about $u = 0$, is a function of k which, in general, does not vanish. The coefficient of $1/(u - k)^{n+1}$, the first term of the principal part of the expansion about $u = k$, is B_n where B_n is one of the expressions (cf. p. 559) whose vanishing is a condition of perspectivity, and the coefficient of $1/(u - k)^n$ is non-vanishing in general. Hence the function has $m + n$ poles, and $m + n$ zeros, and the curves have $m + n$ contacts, the sum of whose parameters is nk. For a fixed k these contacts lie in an involution, $I_{2m-n-1,2n-m+1}$ since $2m - n - 1$ of them will determine E_m and the rest of the $m + n$ contacts.

If $n/2 < m < 2n$ and if $2m - n$ contacts are given, these, substituted in $[d(x\xi)/du]_{v=u} = 0$ will give $2m - n$ homogeneous, linear equations in the $2m - n$ parameters of the family of perspective m-ics. The coefficients are elliptic functions of k, of order $M_j + n + 1$, where M_j is the highest order in k occurring in A_{0j}, A_{1j}, and A_{2j}, where E_m is written

$$\rho\xi_i = \sum_{j=1}^{2m-2} a_i A_{ij}, \qquad (i = 0, 1, 2),$$

and where a_j, $(j = 1, 2, 3, \cdots, 2m - n)$, are the homogeneous parameters. When $m = n = 3$, (cf. p. 570), $M_j = 3$ for each of the three values of j, but its value in general has not been determined. If M is the maximum value of the set M_j, the elimination of a_j gives an elliptic function of order

$$\mu \leqq (2m - n)(M + n + 1).$$

Then μ is the number of sets of $2n - m$ more contacts, i. e., the number of perspective m-ics determined by the $2m - n$ given contacts.

4. *The Family of Cubics Perspective to a Given Cubic.* If in a (1, 1) correspondence between two cubics with three self-corresponding common points, corresponding points are joined, the envelope is a cubic perspective to both. Such a correspondence exists between a cubic and its transform by a collineation with three fixed points on the cubic, and also between a cubic and its transform by a quadratic transformation with the fundamental points and three of the fixed points on the cubic. There are, then, many perspective

cubics associated with a given cubic, and we ask what conditions on the coefficients of the general elliptic cubic envelope E_3 given by

$$\rho\xi_i = \alpha_{i0} + \alpha_{i1}p(v) + \alpha_{i2}p'(v) \qquad (i = 0, 1, 2)$$

will make it perspective to the elliptic cubic C_3, which may be written

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= p(-u+k) \\ \mu x_2 &= p'(-u+k). \end{aligned}$$

The point $u = s$ of C_3 and the line $v = s$ of E_3 are incident for all values of s if and only if, when $v = u$,

$$(x\xi) \equiv x_0\xi_0 + x_1\xi_1 + x_2\xi_2 \equiv 0.$$

This is a sixth order elliptic function with 3rd order poles at $u = 0$ and at $u = k$.

Expanded about $u = 0$, the coördinates of C_3 are

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= p(k) - p'(k)u + p''(k)u^2/2! - \cdots, \\ \mu x_2 &= p'(k) - p''(k)u + p'''(k)u^2/2! - \cdots, \end{aligned}$$

and, for E_3, the expansion about $v = 0$ is

$$\begin{aligned} \rho\xi_i = \alpha_{i0} &+ \alpha_{i1}(1/v^2 + c_2v^2 + c_3v^4 + \cdots) \\ &+ \alpha_{i2}(-2!/v^3 + 2c_2v + 4c_3v^3 + \cdots), \quad (i = 0, 1, 2), \end{aligned}$$

so that the expansion for $(x\xi)_{v=u}$ about $u = 0$ will be

$$\begin{aligned} &(1/u^3)\{-2[\alpha_{02} + \alpha_{12}p(k) + \alpha_{22}p'(k)]\} \\ &+ (1/u^2)\{\alpha_{01} + \alpha_{11}p(k) + \alpha_{21}p'(k) + 2[\alpha_{12}p(k) + \alpha_{22}p'(k)]\} \\ &+ (1/u)\{-[\alpha_{11}p'(k) + \alpha_{21}p''(k) + \alpha_{12}p''(k) + \alpha_{22}p'''(k)]\} \\ &+ \{\alpha_{00} + \alpha_{10}p(k) + \alpha_{20}p'(k) \\ &+ (1/2)[\alpha_{11}p''(k) + \alpha_{21}p'''(k)] + (1/3)[\alpha_{11}p'''(k) + \alpha_{22}p^{\mathrm{IV}}(k)]\} \\ &\qquad + \text{a power series in } u, \end{aligned}$$

or, for brevity,

$$H_3/u^3 + H_2/u^2 + H_1/u + H_0 + \text{a power series in } u.$$

Expanded about $v = k$, $u = k$

$$\begin{aligned} \mu x_0 &= 1 \\ \mu x_1 &= 1/(u-k)^2 + c_2(u-k)^2 + c_3(u-k)^4 + \cdots \\ \mu x_2 &= 2!/(u-k)^3 - 2c_2(u-k) - 4c_3(u-k)^3 - \cdots \end{aligned}$$

and

$$\rho\xi_i = \alpha_{i0} + \alpha_{i1}[p(k) + p'(k)(u-k) + p''(k)(u-k)^2/2! + \cdots]$$
$$+ \alpha_{i2}[p'(k) + p''(k)(u-k) + p'''(k)(u-k)^2/2! + \cdots],$$
$$(i = 0, 1, 2),$$

so that the expansion of $(x\xi)_{v=u}$ is

$$[1/(u-k)^3]\{2[\alpha_{20} + \alpha_{21}p(k) + \alpha_{22}p'(k)]\}$$
$$+ [1/(u-k)^2]\{\alpha_{10} + \alpha_{11}p(k) + \alpha_{12}p'(k) + 2[\alpha_{21}p'(k) + \alpha_{22}p''(k)]\}$$
$$+ [1/(u-k)]\{\alpha_{11}p'(k) + \alpha_{12}p''(k) + \alpha_{21}p''(k) + a_{22}p'''(k)]\}$$
$$+ \{\alpha_{00} + \alpha_{01}p(k) + \alpha_{02}p'(k)$$
$$+ (1/2)[\alpha_{11}p''(k) + \alpha_{12}p'''(k)] + (1/3)[\alpha_{21}p'''(k) + \alpha_{22}p^{\mathrm{IV}}(k)]\}$$
$$+ \text{ a power series in } (u-k).$$

or

$K_3/(u-k)^3 + K_2/(u-k)^2 + K_1/(u-k) + K_0 +$ a power series in $(u-k)$.

Equating to zero the coefficients H_j and K_j $(j = 1, 2, 3)$ of the principal parts of these expansions gives the condition that $(x\xi)_{v=u}$ be a constant, and, by requiring this constant to be zero, we shall have the condition that E_3 be perspective to C_3.

The eight equations

$$H_j = 0, \qquad K_j = 0, \qquad (j = 0, 1, 2, 3),$$

which are linear and homogeneous in α_{ij} are not independent, (cf. p. 559). The sum of the residues is zero. Therefore

$$H_1 + K_1 \equiv 0$$

and we find that the relation

$$(H_0 - K_0) + p(k)(H_2 - K_2) - [p'(k)/2!](H_3 + K_3) \equiv 0$$

holds.

The six equations

(6) $H_0 = 0, \quad H_2 = 0, \quad -H_3/2 = 0, \quad K_2 = 0, \quad K_1 = 0, \quad K_3/2 = 0$

have the matrix

$$\left\| \begin{array}{ccccccccc} 1 & 0 & 0 & p & p''/2 & p'''/3 & p' & p'''/2 & p^{\mathrm{IV}}/3 \\ 0 & 1 & 0 & 0 & p & 2p' & 0 & p' & 2p'' \\ 0 & 0 & 1 & 0 & 0 & p & 0 & 0 & p' \\ 0 & 0 & 0 & 1 & p & p' & 0 & 2p' & 2p'' \\ 0 & 0 & 0 & 0 & p' & p'' & 0 & p'' & p''' \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p & p' \end{array} \right\| .$$

Since $p'(k)$ and $p''(k)$, in the fifth row of the matrix, have no common zeros, the matrix has rank six, so that the equations (6) give six independent homogeneous linear conditions on the nine coefficients α_{ij} of E_3. Therefore, for every value of k, there are ∞^2 class cubics perspective to C_3. We have already shown (p. 562) that k is an independent parameter, so that there are ∞^3 cubics E_3 perspective to C_3.

Solving the equations (6) we find that these perspective cubics are

$$(7) \qquad \rho\xi_i = aA_i(k,v) + bB_i(k,v) + cC_i(k,v), \qquad (i = 0, 1, 2),$$

where

$$a = \alpha_{12} - \alpha_{21}, \qquad b = \alpha_{02} - \alpha_{20}, \qquad c = \alpha_{01} - \alpha_{10},$$

and where

$$\begin{aligned}
A_0(k,v) &= 3g_3p(k) + g_2^2/4 + [g_2p(k) + 3g_3]\, p(v) \\
A_1(k,v) &= g_2p(k) + 3g_3 + g_2p(v) + p'(k)p'(v) \\
A_2(k,v) &= p'(k)p(v) + p'(v) \\
B_0(k,v) &= g_2p(k) + 3g_3 + g_2p(v) - p'(k)p'(v) \\
B_1(k,v) &= g_2 - 12p(k)p(v) \\
B_2(k,v) &= -p'(k) + p'(v) \\
C_0(k,v) &= -p'(k)p(v) - p(k)p'(v) \\
C_1(k,v) &= p'(k) + p'(v) \\
C_2(k,v) &= p(k) - p(v).
\end{aligned}$$

If $a = b = 0$, $c = 1$, E_3 becomes $\rho\xi_i = C_i(k,v)$, i. e., the point

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & p(-k) & p'(-k) \\ 1 & p(v) & p'(v) \end{vmatrix} = 0,$$

which is the point $u = 2k$ of C_3 doubly covered.

These cubics may also be written

$$(7') \qquad \rho\xi_i = l_i(a,b,c,v) + p(k)m_i(a,b,c,v) + p'(k)n_i(a,b,c,v), \qquad (i = 0, 1, 2),$$

where

$$\begin{aligned}
l_0 &= ag_2^2/4 + 3bg_3 + [3ag_3 + bg_2]\, p(v) \\
l_1 &= 3ag_3 + bg_2 + ag_2p(v) + cp'(v) \\
l_2 &= -cp(v) + bp'(v) \\
m_0 &= 3ag_3 + bg_2 + ag_2p(v) - cp'(v) \\
m_1 &= ag_2 - 12bp(v) \\
m_2 &= c - ap'(v)
\end{aligned}$$

$$\begin{aligned}
n_0 &= & cp(v) & \quad + bp'(v) \\
n_1 &= -c & & \quad - ap'(v) \\
n_2 &= \quad b & - ap(v). &
\end{aligned}$$

If, in (7′), we expand in powers of k, multiply throughout by $-k^3/3$ and then put $k=0$ we get

$$\rho\xi_i = n_i(a, b, c, v), \qquad (i = 0, 1, 2),$$

i. e., the point

$$(8) \qquad \begin{vmatrix} x_0 & x_1 & x_2 \\ 1 & p(-v) & p'(-v) \\ a & b & c \end{vmatrix} = 0,$$

or, as a, b and c vary, the ∞^2 points of the plane triply covered.

These triply covered points may also be obtained as follows. Putting $k=0$ in C_3 and writing the coördinates.

$$\mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the incidence relation becomes

$$\alpha_{00} + (\alpha_{10} + \alpha_{01})p(u) + (\alpha_{20} + \alpha_{02})p'(u) \\ + \alpha_{11}p^2(u) + (\alpha_{21} + \alpha_{12})p(u)p'(u) + \alpha_{22}p'^2(u),$$

which vanishes identically when and only when

$$\alpha_{ij} = -\alpha_{ji}$$

and $\qquad \alpha_{ij} = 0 \quad \text{when} \quad i = j,$

or when E_3 is

$$\begin{aligned}
\rho\xi_0 &= & \alpha_{01}p(v) + \alpha_{02}p'(v) &= n_0(\alpha_{12}, \alpha_{02}, \alpha_{01}, v) \\
\rho\xi_1 &= -\alpha_{01} & - \alpha_{12}p'(v) &= n_1(\alpha_{12}, \alpha_{02}, \alpha_{01}, v) \\
\rho\xi_2 &= -\alpha_{02} + \alpha_{12}p(v) & &= n_2(\alpha_{12}, \alpha_{02}, \alpha_{01}, v),
\end{aligned}$$

i. e., when E_3 is one of the points (8).

If two cubics, given respectively by

$$(9) \qquad \rho\xi_i = aA_i(k, -v+k) + bB_i(k, -v+k) + cC_j(k, -v+k)$$

$$(10) \qquad \rho\xi_i = a'A_i(k', -v+k') + b'B_i(k', -v+k') + c'C_i(k', -v+k'), \\ (i = 0, 1, 2),$$

are each perspective to C_3:

$$(11) \qquad \mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the intersection of the line $v=s$ of the one and the line $v=s$ of the other

will give the point $u=s$ of C_3. But this fact is not easy to verify algebraically.

Rewriting (9) and (10) respectively as

$$(9') \qquad \rho\xi_i = a\alpha_i(k,v) + b\beta_i(k,v) + c\gamma_i(k,v) = a\alpha_i + b\beta_i + c\gamma_i$$

and

$$(10') \qquad \rho\xi_i = a'\alpha_i(k',v) + b'\beta_i(k',v) + c'\gamma_i(k',v) = a'\alpha_i' + b'\beta_i' + c'\gamma_i',$$

(where $\alpha_i(k,v) \equiv A_i(k, -v+k)$ and is obtained from it by means of the addition formulae for $p(-v+k)$ and $p'(-v+k)$, and where β_i and γ_i are new forms for B_i and C_i similarly obtained) the generated curve will be

$$(12) \qquad \begin{aligned} \mu x_i = aa' &\begin{vmatrix} \alpha_j & \alpha_j' \\ \alpha_k & \alpha_k' \end{vmatrix} + bb' \begin{vmatrix} \beta_j & \beta_j' \\ \beta_k & \beta_k' \end{vmatrix} + cc' \begin{vmatrix} \gamma_j & \gamma_j' \\ \gamma_k & \gamma_k' \end{vmatrix} \\ + bc' &\begin{vmatrix} \beta_j & \gamma_j' \\ \beta_k & \gamma_k' \end{vmatrix} + ca' \begin{vmatrix} \gamma_j & \alpha_j' \\ \gamma_k & \alpha_k' \end{vmatrix} + ab' \begin{vmatrix} \alpha_j & \beta_j' \\ \alpha_k & \beta_k' \end{vmatrix} \\ - b'c &\begin{vmatrix} \beta_j' & \gamma_j \\ \beta_k' & \gamma_k \end{vmatrix} - c'a \begin{vmatrix} \gamma_j' & \alpha_j \\ \gamma_k' & \alpha_k \end{vmatrix} - a'b \begin{vmatrix} \alpha_j' & \beta_j \\ \alpha_k' & \beta_k \end{vmatrix}, \end{aligned}$$

where $i, j, k = 0, 1, 2$, no two of the three being equal. We shall have proved that this is the curve C_3 if we can show that each of the nine curves obtained by putting equal to zero any four of the six constants a, b, c, a', b', c' is identical with C_3. The work involved in each case is, however, exceedingly troublesome and has been carried through only in a few of the simpler cases when $k=0$ or a proper half period.

When $k=k'$, some of the determinants in (12) vanish and others duplicate each other. Moreover the work can be simplified by writing (9) and (10) as

$$(9'') \qquad \rho\xi_i = aA_i(k,v) + bB_i(k,v) + cC_i(k,v)$$

$$(10'') \qquad \rho\xi_i = a'A_i(k',v) + b'B_i(k',v) + c'C_i(k',v),$$

and writing C_3

$$\mu x_0 = 1, \qquad \mu x_1 = p(-u+k), \qquad \mu x_2 = p'(-u+k).$$

The generated curve will be

$$(13) \qquad x_0 : x_1 : x_2 =$$

$$\begin{vmatrix} M_{A_0} & M_{B_0} & M_{C_0} \\ a & b & c \\ a' & b' & c' \end{vmatrix} : \begin{vmatrix} M_{A_1} & M_{B_1} & M_{C_1} \\ a & b & c \\ a' & b' & c' \end{vmatrix} : \begin{vmatrix} M_{A_2} & M_{B_2} & M_{C_2} \\ a & b & c \\ a' & b' & c' \end{vmatrix},$$

where M_{A_0} is the minor of $A_0(k, v)$ in the matrix

$$(14)\qquad \begin{Vmatrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{Vmatrix}.$$

If

$$(15)\qquad \begin{aligned} M_{A_0} : M_{A_1} : M_{A_2} &= M_{B_0} : M_{B_1} : M_{B_2} \\ &= M_{C_0} : M_{C_1} : M_{C_2} = 1 : p(-u+k) : p'(-u+k), \end{aligned}$$

then

$$x_0 : x_1 : x_2 = 1 : p(-u+k) : p'(-u+k).$$

The verification of (15) can be done by forming proportions of the types

$$(16)\qquad M_{A_1}/M_{A_0} = p(-v+k)/1$$

$$(17)\qquad M_{A_2}/M_{A_1} = p'(-v+k)/p(-v+k)$$

and showing, by means of the identity

$$p'^2(k) = 4p^3(k) - g_2p(k) - g_3$$

that the coefficients of $[p(v)]^j$, $(j = 0, 1, 2, 3, 4, 5)$, and of $p'(v)[p(v)]^i$, $(i = 0, 1, 2)$, vanish, together with the constant terms. This has been done for the proportion (16).

When $k = 0$, the matrix (14) will be

$$\begin{Vmatrix} 0 & -p'(v) & -p(v) \\ p'(v) & 0 & 1 \\ p(v) & -1 & 0 \end{Vmatrix}$$

so that for (15) we have

$$\begin{aligned} 1 : p(v) : -p'(v) &= p(v) : p^2(v) : -p(v)p'(v) \\ &= p'(v) : p(v)p'(v) : -p'^2(v) = 1 : p(-v+0) : p'(-v+0). \end{aligned}$$

For $k =$ a proper half period we have carried through the verification of (15) completely with the help of the identities

$$e_1 + e_2 + e_3 = 0, \quad g_2 = 4(e_1e_2 + e_2e_3 + e_3e_1), \quad g_3 = 4e_1e_2e_3$$

and of

$$\begin{aligned} &4e_1^3 - g_2e_1 - g_3 = p'^2(\omega_1/2) = 0, \\ &12e_1^3 - g_2e_1 = 2g_2e_1 + 3g_3 = 4e_1(2e_1^2 + e_2e_3) \\ &\qquad\qquad = 4e_1(e_1 - e_2)(e_1 - e_3) = 2e_1p''(\omega_1/2), \\ &p(-v + \omega_1/2) = e_1 + (e_1 - e_2)(e_1 - e_3)/[p(v) - e_1], \\ &p'(-v + \omega_1/2) = p'(v)(e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]^2 \end{aligned}$$

and similar expressions involving $\omega_2/2$ and e_3 or $(\omega_1 + \omega_2)/2$ and e_2. We

give a sample of the calculation here because it seems to show a sort of raison d'etre for the complicated functions of g_2, g_3, and k that appear in (7).

When $k = \omega_1/2$,

$$\begin{aligned}
\mu x_0 = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} &= \begin{vmatrix} g_2 - 12 e_1 p(v) & p'(v) \\ p'(v) & e_1 - p(v) \end{vmatrix} \\
&= -\{[p(v) - e_1]\,[g_2 - 12 e_1 p(v)] + p'^2(v)\} \\
&= -4[p(v) - e_1]\{e_1^2 - e_2 e_3 - 3 e_1 p(v) \\
&\quad + [p(v) - e_2]\,[p(v) - e_3]\} \\
&= -4[p(v) - e_1]^3,
\end{aligned}$$

$$\begin{aligned}
\mu x_1 = \begin{vmatrix} B_2 & C_2 \\ B_0 & C_0 \end{vmatrix} &= \begin{vmatrix} p'(v) & e_1 - p(v) \\ g_2 e_1 + 3 g_3 + g_2 p(v) & -e_1 p'(v) \end{vmatrix} \\
&= e_1[4p^3(v) - g_2 p(v) - g_3] \\
&\quad - g_2 e_1^2 - 3 g_3 e_1 + 3 g_3 p(v) + g_2 p^2(v) \\
&= -4 e_1 p^3(v) + g_2[p^2(v) + e_1 p(v) - e_1^2] \\
&\quad + g_3[3p(v) - 2e_1] \\
&= -4\{e_1 p^3(v) + (e_2 e_3 - e_1^2)[p^2(v) + e_1 p(v) - e_1^2] \\
&\quad - e_1 e_2 e_3[3p(v) - 2e_1]\} \\
&= -4\{e_1[p(v) - e_1]^3 + (2e_1^2 + e_2 e_3)[p(v) - e_1]^2\} \\
&= -4[p(v) - e_1]^3\{e_1 + (e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]\} \\
&= -4[p(v) - e_1]^3 p(-v + \omega_1/2),
\end{aligned}$$

$$\begin{aligned}
\mu x_2 = \begin{vmatrix} B_0 & C_0 \\ B_1 & C_1 \end{vmatrix} &= \begin{vmatrix} g_2 e_1 + 3 g_3 + g_2 p(v) & -e_1 p'(v) \\ g_2 - 12 e_1 p(v) & p'(v) \end{vmatrix} \\
&= p'(v)\{2 g_2 e_1 + 3 g_3 - (12 e_1^2 - g_2) p(v)\} \\
&= p'(v)\{4 e_1 (2 e_1^2 + e_2 e_3) - 4(e_1^2 + e_2 e_3) p(v)\} \\
&= -4[p(v) - e_1]^3 p'(v)(e_1 - e_2)(e_1 - e_3)/[p(v) - e_1]^2 \\
&= -4[p(v) - e_1]^3 p'(-v + \omega_1/2).
\end{aligned}$$

5. *The Contacts of Perspective Cubics.* Two perspective cubics have 6 contacts, by Theorem II, p. 563. If k is known and C_3 given, two contacts determine the ratios $a : b : c$ in (7), and so fix E_3 and the other four contacts.

If three contacts are given,

$$u = u_i, \qquad (i = 1, 2, 3),$$

then $[d(x\xi)/du]_{v=u=u_i}$ gives the three equations

$$\begin{aligned}
&[a g_2 p(k) + 3 g_3 + b g_2 - c p'(k)]\,p'(u_i - k) + [a g_2 - 12 b p(k)]\,p(u_i)\,p'(u_i - k) \\
&\quad + [a p'(k) + c]\,p'(u_i)\,p'(u_i - k) + [b p'(k) + c p(k)]\,p''(u_i - k) \\
&\quad + [a p'(k) - c]\,p(u_i)\,p''(u_i - k) + [-a p(k) + b]\,p'(u_i)\,p''(u_i - k).
\end{aligned}$$

We have underlined the terms of highest order in k occurring in the coefficient of a of b or of c, showing that the coefficients of a and of b are 7th order elliptic functions in k, while that of c is of the 6th order. Elimination of a, b, c gives, therefore, an elliptic function of order 20 in k. This means that, when three contacts are given, twenty sets of three more contacts, or twenty perspective cubics, are determined.

6. *Perspective Cubics Associated with Collineation and with Quadratic Transformation.* We naturally wish to compare the set of cubics (7) with the sets obtained by joining the points of C_3 with the corresponding points of its transforms by a quadratic transformation with fundamental points and three fixed points on C_3 and by a collineation with fixed points on C_3. An expression for the cubics in the latter case has been obtained in terms of sigma functions and it is shown that they are ∞^3 in number. The method is as follows.

Putting on the collineation the condition that it leave fixed the lines joining the fixed points u_0, u_1 and u_2 of C_3, which is written

$$\mu x_0 = 1, \qquad \mu x_1 = p(u), \qquad \mu x_2 = p'(u),$$

the desired envelope is expressed in terms of D_i and certain minors of D_i, where D_i is the determinant obtained from

$$D = \begin{vmatrix} 1 & p(u_0) & p'(u_0) \\ 1 & p(u_1) & p'(u_1) \\ 1 & p(u_2) & p'(u_2) \end{vmatrix}$$

by substituting the row 1, $p(u)$, $p'(u)$ for the row 1, $p(u_i)$ $p'(u_i)$.

Expressions for D_i and for the minor

$$\begin{vmatrix} 1 & p(u) \\ 1 & p(u_i) \end{vmatrix}$$

in terms of sigma functions are well known. The other minors needed were obtained by putting first $u_j = \omega_1/2$ and then $u_j = \omega_2/2$ $(j \neq i)$ in D_i and solving simultaneously the two equations so obtained. In this way the perspective envelopes can be expressed, in a rather unpromising manner, in terms of sigma functions. They involve, besides u_0, u_1 and u_2 three homogeneous parameters, and, by setting two of the latter equal to zero it is easy to show that there are just three effective parameters.

In the case of quadratic transformation, by considering the condition on A_1, A_2, and A_3, the fundamental points, and on B_1, B_2, and B_3, the fixed points on C_3, that A_1B_1, A_2B_2, A_3B_3 be concurrent, a proof is obtained that

there are ∞^4 quadratic transformations of this type. By noting that the points of C_3 which lie on a conic through A_1, A_2 and A_3 will be collinear on the transform of C_3, we obtain k in terms of the parameters of A_1, A_2, and A_3. This makes it necessary to exclude from among the permissible values of k, those values for which A_1, A_2, and A_3 (or B_1, B_2, and B_3) are collinear. We find, in fact, that $k = 0$ or a proper third period cannot be associated with curves obtained in this manner.

BIBLIOGRAPHY.

[1] A. Brill, "Ueber rationalen Curven und Regelflächen," *Mathematische Annalen*, Vol. 36 (1889), pp. 230-238.

[2] A. B. Coble, "Geometric Aspects of the Abelian Modular Functions of Genus Four," Part I, *American Journal of Mathematics*, Vol. 46, (1924), pp. 143-192, 5.

[3] ———, "Geometric Aspects of the Abelian Modular Functions of Genus Four," Part II, *American Journal of Mathematics*, Vol. 51 (1929), pp. 495-514, 11.

[4] ———, "Symmetric Binary Forms and Involutions," *American Journal of Mathematics*, Vol. 32 (1910), pp. 333-364. 15.

[5] J. C. F. Haase, "Zur Theorie der ebenen Curven vierter Ordnung mit $(n-1)(n-2)/2$ Doppel-und-Rückkehrpunkten," *Mathematische Annalen*, Vol. 2 (1870), pp. 515-548.

[6] St. Jolles, "Die Théorie des Osculanten und des Sehnensystems der Raumcurve IV Ordnung II Species," *Inaugural Dissertation Strassburg*, 1-19 (1883).

[7] Fr. Meyer, "Ueber die mit der Erzeugung der Raumcurven vierter Ordnung Zweiter Species verknüpften algebraischen Processe," *Mathematische Annalen*, Vol. 29 (1887), pp. 447-467.

[8] ———, "Zur Erzeugung der rationalen Curven," *Mathematische Naturwissenschaftliche Mitteilungen herausgegeben* von Dr. Otto Böklen, Tübingen II_1, (1889), pp. 33-38.

[9] ———, "Zur algebraischen Erzeugung sämmtlicher auch der zerfallenden ebenen rationalen Curven vierter Ordnung," *Mathematische Annalen*, Vol. 31 (1888), pp. 96-133.

[10] ———, "Zur Théorie der reduciblen gauzen Functionen von n Variablen," *Mathematische Annalen*, Vol. 30 (1887), pp. 30-74.

[11] H. Schroters, "Ueber die Erzeugnisse krummer projektivischer Gebilde," *Crelle's Journal*, Vol. 54 (1857), pp. 31-47.

[12] R. Schumacher, "Zur Eintheilung der Strahlencongruenzen zweiter Ordnung mit Brenn-oder-singulären Linien; Ebenbüschel zweiter Ordnung in perspectiver Lage zu rationalen Curven," *Mathematische Annalen*, Vol. 30 (1887).

[13] C. Segre, "Récherches générales sur les courbes et les surfaces réglées algébrigues. I Partie," *Mathematische Annalen*, Vol. 30 (1887), pp. 203-226.

[14] ———, "Récherches générales sur les courbes et les surfaces réglées algébriques. II Partie," *Mathematische Annalen*, Vol. 34 (1889), pp. 1-25.

[15] W. Stahl, "Zur Erzeugung der ebenen rationalen Curven," *Mathematische Annalen*, Vol. 38 (1891), pp. 561-585.

[16] ———, "Zur Erzeugung der rationalen Raumcurven," *Mathematische Annalen*, Vol. 40 (1892), pp. 1-54.

[17] E. Study, "Ueber die rationale Curven vierter Ordnung," *Leipzige Berichte*, Vol. 38 (1886), pp. 3-9.

CONJUGATE NETS AND THE LINES OF CURVATURE.

By Ernest P. Lane.

1. *Introduction.* The purpose of this paper is twofold; first, to make some contributions to the projective differential geometry of a general conjugate net on an analytic surface in ordinary space; second, to connect this geometry with the metric differential geometry of the lines of curvature.

Two sections are concerned with purely projective geometry. An analytic basis for the projective differential theory of a parametric conjugate net on a surface in ordinary space is established in Section 2, where the parts of this theory that are essential for what follows are summarized. In Section 3, as one of the principal contributions of this paper, two quadrics called *conjugate osculating quadrics* are associated with each point of a given curve on the surface sustaining a given conjugate net. The reader who is familiar with the *asymptotic osculating quadrics,* defined by Bompiani and Klobouček, will observe that the definitions of the two kinds of quadrics are quite similar. An asymptotic osculating quadric at a point of a curve on a surface is determined by three consecutive asymptotic tangents of one family, whereas in defining a conjugate osculating quadric we shall use instead three consecutive tangents of the curves of one family of a given conjugate net. The equations of these quadrics will be derived and their properties briefly studied.

The last three sections are more or less metric in character. Section 4 is taken up with a summary of the elements of the metric differential geometry of the lines of curvature on a surface, preparatory to what follows. In Section 5, as a second contribution of this paper, a transformation is devised, *from* the local projective homogeneous coördinates based on a certain projectively covariant tetrahedron at a general point of a surface, which are used in the projective theory of a general conjugate net, *to* the local cartesian non-homogeneous coördinates based on a certain metrically covariant trihedron at a general point of a surface, which are used in the metric theory of the lines of curvature. This transformation makes it possible to study in Section 6 for the particular metrically defined conjugate net called the lines of curvature various configurations that are considered in the projective theory of a general conjugate net.

2. *Conjugate Nets.* In this section we establish an analytic basis for

the projective differential geometry of a parametric conjugate net on a surface in ordinary space. Some portions of this geometry, which will be needed later on, are summarized. The union curves of the axis congruence of a conjugate net appear. The equations of the ray-point cubic and ray conic of a pencil of conjugate nets are written, as well as the equation of the quadric of Lie. By *net* we shall mean *conjugate net* unless otherwise indicated.

The projective differential geometry of conjugate nets was studied by G. M. Green, who based his theory * on a system of differential equations of the form

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \\ y_{uv} &= \phantom{ay_{vv} + {}} b'y_u + c'y_v + d'y. \end{aligned} \tag{1}$$

The obvious lack of symmetry in these equations has led Slotnick to use † a different system. We shall use here a system which differs only notationally and in the choice of proportionality factor from that of Slotnick.

Let the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x on a surface S referred to a conjugate net N_x in ordinary space be given as analytic functions of two independent variables u, v. The axis at the point P_x of the net N_x is the line of intersection of the osculating planes of the parametric curves C_u, C_v at P_x. Let P_y be the point which is the harmonic conjugate of P_x with respect to the two foci of the axis regarded as generating a congruence (the axis congruence) when the point P_x varies over the surface S. Then x and y satisfy a system of equations of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0). \end{aligned} \tag{2}$$

We shall use this system as the basis of our projective theory.

From the equations

$$(x_{vv})_u = (x_{uv})_v, \qquad (x_{uu})_v = (x_{uv})_u$$

we obtain

$$y_u = fx - nx_u + sx_v + Ay, \qquad y_v = gx + tx_u + nx_v + By, \tag{3}$$

where we have placed

* Green, "Projective Differential Geometry of One-Parameter Families of Curves, and Conjugate Nets on a Curved Surface," First Memoir, *American Journal of Mathematics*, Vol. 37 (1915), p. 215; Second Memoir, Vol. 38 (1916), p. 287.

† Slotnick, "On the Projective Differential Geometry of Conjugate Nets," *American Journal of Mathematics*, Vol. 53 (1931), p. 143.

$$
(4)\quad
\begin{aligned}
fN &= c_v + ac + bq - c\delta - qu, & gL &= c_u + bc + ap - c\alpha - p_v,\\
-nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v,\\
sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p,\\
A &= b - (\log N)_u, & B &= a - (\log L)_v.
\end{aligned}
$$

From the equation $(y_u)_v = (y_v)_u$ we could obtain in light of the fact that the four points x, x_u, x_v, y are not coplanar, four integrability conditions, which we do not need to write here.

The ray-points, or Laplace transformed points, ρ, σ of the curves C_u, C_v respectively at the point P_x are defined by the formulas

$$
(5)\qquad \sigma = x_v - ax, \qquad \rho = x_u - bx.
$$

The Laplace-Darboux point invariants H, K, the tangential invariants $\boldsymbol{H}, \boldsymbol{K}$, the Weingarten invariants $W^{(u)}, W^{(v)}$, the invariants $\mathfrak{B}', \mathfrak{C}', \mathfrak{D}$ of Green, and the invariant r of Eisenhart * are expressed in terms of the coefficients of systems (2) by the formulas

$$
(6)\quad
\begin{aligned}
H &= c + ab - a_u, & K &= c + ab - b_v,\\
\boldsymbol{H} &= sN, & \boldsymbol{K} &= tL,\\
W^{(u)} &= \boldsymbol{H} - K, & W^{(v)} &= \boldsymbol{K} - H,\\
8\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & 8\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u,\\
\mathfrak{D} &= -2nL, & r &= N/L.
\end{aligned}
$$

The invariant r is the reciprocal of Green's invariant a.

We shall have occasion to use the covariant tetrahedron x, ρ, σ, y as a local tetrahedron of reference with a unit point chosen so that a point

$$
x_1 x + x_2 \rho + x_3 \sigma + x_4 y
$$

has local coördinates proportional to $x_1, \cdots, x_4$. In this coördinate system the equations of the osculating planes of the curves C_u, C_v at the point P_x are respectively $x_3 = 0$ and $x_2 = 0$. The equation of the osculating plane of the curve C_λ of the family defined by the equation

$$
(7)\qquad dv - \lambda du = 0
$$

at the point P_x is found by making use of the fact that this plane is determined by the points x, x', x'', where

$$
(8)\qquad x' = x_u + x_v\lambda, \quad x'' = x_{uu} + 2x_{uv}\lambda + x_{vv}\lambda^2 + x_v\lambda' \qquad (\lambda' = \lambda_u + \lambda\lambda_v).
$$

The result can be written in the form

$$
(9)\qquad (L + N\lambda^2)(x_3 - \lambda x_2) - \lambda\,[4(\mathfrak{C}' - \lambda\mathfrak{B}') + (\log \lambda r^{1/2})']\, x_4 = 0,
$$

* Eisenhart, *Transformations of Surfaces*, Princeton University Press, p. 101.

wherein $\mathfrak{B}'$, $\mathfrak{C}'$ are two invariants appearing in equations (6). This plane coincides with the tangent plane, $x_4 = 0$, at each point of C_λ if, and only if, C_λ is an asymptotic curve; in this case λ satisfies the equation

$$L + N\lambda^2 = 0. \tag{10}$$

The union curves of the axis congruence of a net are those curves such that at each point of one of them its osculating plane contains the axis through the point. The curve C_λ is a union curve of the axis congruence of the net N_x in case the coefficient of x_4 in equation (9) vanishes. This condition can be written in the form

$$\lambda' = -[4\mathfrak{C}' + (\log r^{1/2})_u]\lambda + [4\mathfrak{B}' - (\log r^{1/2})_v]\lambda^2. \tag{11}$$

If λ is replaced by dv/du this equation becomes an equation of the second order for v as a function of u along a union curve. So we obtain the well-known result that there is a two-parameter family of union curves of a given axis congruence. The fact that equation (11) is an equation of Riccati for λ leads to the remark that the cross ratio of four particular solutions is constant, and hence to the following theorem, apparently unobserved before and capable of generalization to any two-parameter family of hypergeodesics containing the parametric curves on a surface:

Any four one-parameter families of union curves of the axis congruence of a conjugate net on a surface in ordinary space have the property that the cross ratio of the tangents of the four curves of the families at a point of the surface is the same at every point of the surface.

Replacing λ by dv/du in equation (10) we obtain *the differential equation of the asymptotic curves* on the surface S, namely,

$$L du^2 + N dv^2 = 0. \tag{12}$$

In order that the differential equation

$$(dv - \lambda du)(dv - \mu du) = 0 \tag{13}$$

may represent a conjugate net the two directions defined by this equation must separate harmonically the two asymptotic directions satisfying equation (12). A condition necessary and sufficient therefor is the following, which we shall suppose satisfied whenever we employ the function μ hereinafter:

$$\mu = -1/\lambda r \qquad (r = N/L). \tag{14}$$

The two curves of such a conjugate net, at a point of a surface, may be designeted as C_λ, C_μ respectively.

The ray-points ρ_λ, σ_μ of the curves C_λ, C_μ at a point P_x can be shown * to be represented by the formulas

$$(15)\qquad \begin{aligned}\rho_\lambda &= (1+\lambda^2 r)(\rho+\mu\sigma) - [\lambda P - Q + (\log \lambda^2 r)']\,x,\\ \sigma_\mu &= (1+\lambda^2 r)(\rho+\lambda\sigma) + \lambda(P+\lambda r Q)x,\end{aligned}$$

wherein P, Q are defined by placing

$$(16)\qquad P = 4\mathfrak{B}' - (\log \lambda r^{1/2})_v, \quad Q = 4\mathfrak{C}' + (\log \lambda r^{1/2})_u.$$

The equations of *the ray-point cubic* of the pencil of conjugate nets determined by the conjugate net (13) at a point P_x are †

$$(17)\qquad \begin{aligned}x_4 = (x_2^2 + r x_3^2)(x_1 - \mathfrak{C}'x_2 - \mathfrak{B}'x_3)&\\ + \mathfrak{C}'x_2^3 - 3\mathfrak{B}'x_2^2 x_3 - 3r\mathfrak{C}'x_2 x_3^2 + r\mathfrak{B}'x_3^3 &= 0,\end{aligned}$$

and the equations of *the ray conic* of the pencil are

$$(18)\qquad x_4 = (\mathfrak{B}'^2 + r\mathfrak{C}'^2)(x_2^2 + r x_3^2) - r(x_1 - \mathfrak{C}'x_2 - \mathfrak{B}'x_3)^2 = 0.$$

The equation of any quadric of Darboux at a point P_x of a surface S referred to a conjugate net N_x is

$$(19)\qquad L x_2^2 + N x_3^2 + x_4(-2x_1 + 4\mathfrak{C}'x_2 + 4\mathfrak{B}'x_3 + k x_4) = 0,$$

where k is arbitrary. For *the quadric of Lie* the value of k is given ‡ by

$$(20)\qquad LNk = 2(L\mathfrak{B}'^2 + N\mathfrak{C}'^2) + L\mathfrak{B}'(\log \mathfrak{B}'r^{1/2})_v + N\mathfrak{C}'(\log \mathfrak{C}'/r^{1/2})_u.$$

The entire theory of surfaces in ordinary space can, of course, be developed on the basis of a parametric conjugate net, in spite of the fact that it may ordinarily be more convenient to take the asymptotic curves as parametric. For some further developments in this direction the reader may consult the last two references cited.

3. *The Conjugate Osculating Quadrics.* In order to formulate a definition let us consider a conjugate net N on a surface S in ordinary space, and on S any curve C not belonging to N. At a point P of the curve C, and at two neighboring points P_1, P_2 on C, let us construct the tangents of the curves of one family of the net N. These three tangents determine a quadric, and the limit of this quadric as the points P_1, P_2 approach P along C is a *conjugate osculating quadric* at the point P of the curve C on the net N. The other

* Lane, "Bundles and Pencils of Nets on a Surface," *Transactions of the American Mathematical Society*, Vol. 28 (1926), p. 161.

† Lane, *loc. cit.*, p. 163.

‡ Miss Hagen, Chicago Master's Thesis, 1926, p. 12.

conjugate osculating quadric at P of C on N is defined similarly by using tangents of the other family of the net N. The first problem is to find the equations of the quadrics just defined. Then their geometrical properties can be investigated analytically.

We now set out to find the equation of the conjugate osculating quadric Q_u determined by three consecutive u-tangents at a point P_x of a curve C_λ of the family represented by equation (7) on the parametric net N_x on an integral surface S of system (2). We should like eventually to have this equation referred to the covariant tetrahedron x, ρ, σ, y. From system (2) by differentiation and substitution one obtains

$$\begin{aligned} x_{uuu} &= (p_u + \alpha p + fL)x + (\alpha_u + \alpha^2 + p - nL)x_u \\ &\quad + sLx_v + (L_u + \alpha L + AL)y, \\ x_{uuv} &= (c_u + bc + ap)x + (a_u + c + ab + a\alpha)x_u + (b_u + b^2)x_v + aLy, \\ x_{uvv} &= (c_v + ac + bq)x + (a_v + a^2)x_u + (b_v + c + ab + b\delta)x_v + bNy. \end{aligned} \tag{21}$$

Similarly any derivative of x can be expressed as a linear combination of x, x_u, x_v, y. Any point X on the curve C_λ and near the point P_x can be defined by a power series in the increment Δu corresponding to displacement on C_λ from P_x to the point X, of the form

$$X = x + x'\Delta u + x''\Delta u^2/2 + \cdots.$$

Then we find

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 y,$$

where

$$\begin{aligned} &x_1 = 1 + \cdots, \qquad x_2 = \Delta u + \cdots, \\ &x_3 = \lambda\Delta u + (\lambda' + 2b\lambda + \delta\lambda^2)\Delta u^2/2 + \cdots, \\ &x_4 = (L + N\lambda^2)\Delta u^2/2 + \cdots. \end{aligned} \tag{22}$$

These series represent the local coördinates $x_1, \cdots, x_4$ of the point X, referred to the tetrahedron x, x_u, x_v, y with suitably chosen unit point, to terms of as high degree in Δu as will be needed in this paper. Similarly, expanding X_u, we have

$$\begin{aligned} X_u &= x_u + (x_u)'\Delta u + (x_u)''\Delta u^2/2 + \cdots \\ &= x_1 x + x_2 x_u + x_3 x_v + x_4 y, \end{aligned}$$

where

$$\begin{aligned} &x_1 = (p + c\lambda)\Delta u + \cdots, \qquad x_2 = 1 + (\alpha + a\lambda)\Delta u + \cdots, \\ &x_3 = b\lambda\Delta u + [sL + 2(b_u + b^2)\lambda \\ &\qquad + (b_v + c + ab + b\delta)\lambda^2 + b\lambda']\,\Delta u^2/2 + \cdots, \\ &x_4 = L\Delta u + (L_u + \alpha L + AL + 2aL\lambda + bN\lambda^2)\Delta u^2/2 + \cdots. \end{aligned} \tag{23}$$

In order to calculate power series expansions for the local coördinates $x_1, \cdots, x_4$ of any point $hX + kX_u$ on the u-tangent XX_u at the point X, it is sufficient to multiply the series (22) by h and the series (23) by k and add corresponding series. Then writing the equation of the most general quadric surface and demanding that it be satisfied by the power series thus calculated, identically in h, k and in Δu as far as the terms of the second degree, we obtain *the equation of the quadric Q_u referred to the tetrahedron x, x_u, x_v, y*, namely,

$$Dx_3^2 + Ex_3x_4 + Fx_4^2 - G[x_2x_3 - \lambda x_4(x_1 + bx_2)/L] = 0, \tag{24}$$

the ratios of the coefficients D, E, F, G being defined by

$$\begin{aligned} 2\lambda LD &= G(L - N\lambda^2), \\ 2LE &= G[\alpha + \delta\lambda - b - A - (\log L) + (\log \lambda)'], \\ L^2F &= G\lambda[b_u + b^2 - b\alpha - p + (\lambda/2)(b_v - c - ab + sL/\lambda^2)]. \end{aligned} \tag{25}$$

For the purpose of writing the equation of the quadric Q_u referred to the tetrahedron x, ρ, σ, y a simple computation shows that it is sufficient to replace x_1 in equation (24) by $x_1 - bx_2 - ax_3$. Making this substitution and simplifying the coefficients by means of equations (4), (6), we arrive at the desired *equation of the conjugate osculating quadric Q_u, referred to the tetrahedron x, ρ, σ, y*, namely,

$$\begin{aligned} (L - N\lambda^2)x_3^2 - \lambda[4(\mathfrak{C}' + \lambda\mathfrak{B}') - (\log \lambda r^{1/2})']x_3x_4 \\ + \lambda(2n\lambda - \lambda^2K/L + \mathbf{H}/N)x_4^2 + 2\lambda(\lambda x_1x_4 - Lx_2x_3) = 0. \end{aligned} \tag{26}$$

The equation of the conjugate osculating quadric Q_v at the point P_x of the curve C_λ on the net N_x can be written by interchanging u and v and making the appropriate symmetrical interchanges of the other symbols. The result is

$$\begin{aligned} (L - N\lambda^2)x_2^2 + [4(\mathfrak{C}' + \lambda\mathfrak{B}') + (\log \lambda r^{1/2})']x_2x_4 \\ + (2n + H/\lambda N - \lambda\mathbf{K}/L)x_4^2 - 2(x_1x_4 - \lambda Nx_2x_3) = 0. \end{aligned} \tag{27}$$

Several interesting theorems can be easily established. For example, the tangent plane, $x_4 = 0$, intersects the quadric Q_u in the u-tangent, $x_4 = x_3 = 0$, of course, and also in the residual line

$$x_4 = (L - N\lambda^2)x_3 - 2\lambda Lx_2 = 0. \tag{28}$$

Similarly, the tangent plane intersects Q_v in the v-tangent, $x_4 = x_2 = 0$, and in the residual line

$$x_4 = (L - N\lambda^2)x_2 + 2\lambda Nx_3 = 0. \tag{29}$$

The line (28) coincides with the v-tangent,—and the line (29) with the u-tangent,—if, and only if, the curve C_λ is such that

$$L - N\lambda^2 = 0. \tag{30}$$

If this happens at every point of C_λ, then C_λ is a curve of *the associate conjugate net* of the parametric net N_x, that is, the conjugate net whose tangents at each point of the surface S separate the tangents of N_x harmonically. So we have proved the theorem:

Each of the two conjugate osculating quadrics at every point of a curve on a net intersects the tangent plane of the net in the tangents of the net if, and only if, the curve belongs to the associate conjugate net.

When $L - N\lambda^2 \neq 0$ the two lines (28), (29) coincide at every point of a curve C_λ if, and only if, $L + N\lambda^2 = 0$. But in this case the curve C_λ is an asymptotic curve. Moreover, in this case the two lines coincide with the tangent, $x_4 = x_3 - \lambda x_2 = 0$, of the curve C_λ. Hence we have the theorem:

The two residual lines in which the tangent plane intersects the two conjugate osculating quadrics at every point of a curve on a net coincide if, and only if, the curve is an asymptotic curve on the surface sustaining the net; then the lines coincide with the tangent of the curve.

When $(L - N\lambda^2)(L + N\lambda^2) \neq 0$, the cross ratio of the two tangents of the net and the two residual lines (28), (29), in one of the possible orders, is

$$(L + N\lambda^2)^2/(L - N\lambda^2)^2.$$

The intersections of the quadrics Q_u, Q_v with the other three faces of the covariant tetrahedron of reference are of some interest, but we shall not prolong these considerations to include the details beyond mentioning the following facts.

The osculating plane, $x_3 = 0$, at a point P_x of a curve C touches the corresponding quadric Q_u of a curve C_λ in the ray-point ρ of the curve C_v. One of the generators in the plane $x_3 = 0$ is the u-tangent, $x_3 = x_4 = 0$. The other coincides with the line $x_3 = x_1 = 0$ if, and only if,

$$2n\lambda - \lambda^2 K/L + \mathbf{H}/N = 0.$$

The equation of *the cone projecting from the point P_x the curve of intersection of the two quadrics Q_u, Q_v* is obtained by eliminating x_1 from equations (26), (27). The result is

$$(31)\qquad \begin{aligned}&(L - N\lambda^2)(x_3 - \lambda x_2)^2 - \lambda[4(\mathfrak{C}' + \lambda\mathfrak{B}') - (\log \lambda r^{1/2})']x_3x_4\\ &+ \lambda^2[4(\mathfrak{C}' + \lambda\mathfrak{B}') + (\log \lambda r^{1/2})']x_2x_4\\ &+ \lambda[4n\lambda - \lambda^2(K + \mathbf{K})/L + (H + \mathbf{H})/N]x_4^2 = 0.\end{aligned}$$

This cone is clearly tangent to the tangent plane, $x_4 = 0$, along the tangent line, $x_4 = x_3 - \lambda x_2 = 0$, of the curve C_λ. Its discriminant vanishes, so that it is a pair of planes and the intersection is two conics, in case

$$(32)\qquad (L - N\lambda^2)(\lambda^2 r)' = 0.$$

When the first factor vanishes one of the planes is the tangent plane, $x_4 = 0$, and the conic in it is composed of the tangents of the net N_x, as we have already seen. The equation of the other plane can easily be read from equation (31). When the second factor vanishes the product $\lambda^2 r$ is *constant along the curve* C_λ, and λ satisfies the equation

$$(33)\qquad \lambda' = -\lambda(\log r^{1/2})_u - \lambda^2(\log r^{1/2})_v.$$

Evidently this equation is satisfied if $\lambda^2 r = \text{const.}$ *over the surface.* Then the family (7) is the most general family such that at each point of the surface its tangent and either one of the associate conjugate tangents form with the tangents of the net N_x a cross ratio which is the same for every point of the surface. The equations of the two planes when the second factor of equation (32) vanishes are easily obtained by factoring the left member of equation (31), and it may be observed that these planes intersect in the tangent of the curve C_λ.

The relations of two conjugate osculating quadrics Q_u for the curves of two conjugate families are of some interest. The equation of the quadric Q_u for the curve C_μ, with μ satisfying equation (14), can be written at once by replacing λ by $-1/r\lambda$ in equation (26). The result, after some simplification, is

$$(34)\qquad \begin{aligned}&r\lambda(L - N\lambda^2)x_3^2 + \lambda[4(\mathfrak{B}' - r\lambda\mathfrak{C}') + (\log \lambda r^{1/2})_v - r\lambda(\log \lambda r^{1/2})_u]x_3x_4\\ &-(2n\lambda + K/N - r\lambda^2\mathbf{H}/N)x_4^2 - 2\lambda(x_1x_4 + N\lambda x_2x_3) = 0.\end{aligned}$$

Eliminating x_1 between equations (26), (34) we obtain the equation of *the cone projecting from the point P_x the curve of intersection of the quadrics Q_u for the curves of two conjugate families,* namely,

$$(35)\qquad \begin{aligned}&(1 + r\lambda^2)[(L - N\lambda^2)x_3^2 - 4\lambda\mathfrak{C}'x_3x_4 + \lambda W^{(u)}x_4^2/N - 2\lambda Lx_2x_3]\\ &+ \lambda[(1 - r\lambda^2)(\log \lambda r^{1/2})_u + 2\lambda(\log \lambda r^{1/2})_v]x_3x_4 = 0.\end{aligned}$$

As is geometrically obvious, the cone (35) is indeterminate, so that the quadrics (26), (34) coincide, in case $1 + r\lambda^2 = 0$, that is, in case the curve

C_λ is an asymptotic curve. Otherwise, the cone (35) is a pair of planes, so that the quadrics intersect in two conics, in case $W^{(u)} = 0$, that is, in case the u-curves of the net N_x form a W congruence. Then one of the planes is the plane $x_3 = 0$, and the equation of the other is easily read from equation (35). If $W^{(u)} \neq 0$, the cone (35) is tangent to the osculating plane, $x_3 = 0$, of the u-curve, touching it along its tangent line, $x_3 = x_4 = 0$.

4. *The Local Trihedron.* In the metric differential geometry of a surface in ordinary space it is convenient for some purposes to take the lines of curvature for the parametric curves and to employ a local trihedron at a point of the surface, whose edges are the tangents of the lines of curvature and the normal at the point of the surface. This section is designed to introduce these conceptions and to collect some formulas which will be used later on in this paper.

Let us consider in ordinary metric space a surface whose parametric equations in cartesian coordinates are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v). \tag{36}$$

Let the lines of curvature be the parametric curves on this surface. Then its first and second fundamental forms, written in the customary notation, are

$$Edu^2 + Gdv^2 \qquad Ddu^2 + D''dv^2. \tag{37}$$

The differential equations satisfied by the coördinates x, y, z of a variable point on the surface and by the direction cosines X, Y, Z of the normal of the surface at the point take the form

$$\begin{aligned} x_{uu} &= (\log E^{1/2})_u x_u - (E_v/2G)x_v + DX, \\ x_{uv} &= (\log E^{1/2})_v x_u + (\log G^{1/2})_u x_v, \\ x_{vv} &= -(G_u/2E)x_u + (\log G^{1/2})_v x_v + D''X, \\ X_u &= -x_u/R_1, \qquad X_v = -x_v/R_2, \end{aligned} \tag{38}$$

where *the principal radii of normal curvature* R_1, R_2 at a point of the surface are defined by the formulas

$$R_1 = E/D, \qquad R_2 = G/D''. \tag{39}$$

The corresponding *radii of geodesic curvature* ρ_1, ρ_2 of the u-curve and of the v-curve respectively at the point are given * by the formulas

$$1/\rho_1 = -(\log E)_v/2G^{1/2}, \qquad 1/\rho_2 = (\log G)_u/2E^{1/2}. \tag{40}$$

* Eisenhart, *Differential Geometry*, Ginn and Co., 1909, p. 134.

The three integrability conditions of equations (38) are *the equation of Gauss*

$$-2(EG)^{1/2}K_t=[G_u/(EG)^{1/2}]_u+[E_v/(EG)^{1/2}]_v, \tag{41}$$

wherein the total curvature K_t is defined by

$$K_t=1/R_1R_2, \tag{42}$$

and *the two equations of Codazzi,*

$$\begin{aligned}(1/R_1)_v&=(1/R_2-1/R_1)(\log E^{1/2})_v,\\ (1/R_2)_u&=(1/R_1-1/R_2)(\log G^{1/2})_u.\end{aligned} \tag{43}$$

As a local trihedron of reference at a point (x, y, z) of the surface, we shall take the origin at this point, the ξ-axis along the u-tangent, the η-axis along the v-tangent, and the ζ-axis along the normal. If $\bar{x}, \bar{y}, \bar{z}$ are the general coördinates of a point having local coördinates ξ, η, ζ, the equations of transformation between the general and local cartesian systems can be written in the form

$$\begin{aligned}\bar{x}-x&=\xi x_u/E^{1/2}+\eta x_v/G^{1/2}+X\zeta,\\ \bar{y}-y&=\xi y_u/E^{1/2}+\eta y_v/G^{1/2}+Y\zeta,\\ \bar{z}-z&=\xi z_u/E^{1/2}+\eta z_v/G^{1/2}+Z\zeta.\end{aligned} \tag{44}$$

The equations of the inverse transformation are

$$\begin{aligned}\xi&=(\bar{x}-x,\quad x_v/G^{1/2},\quad X),\\ \eta&=(\bar{x}-x,\quad X,\quad x_u/E^{1/2}),\\ \zeta&=(\bar{x}-x,\quad x_u/E^{1/2},\quad x_v/G^{1/2}),\end{aligned} \tag{45}$$

parentheses indicating determinants of which only a typical row is written.

5. *Transformation of Coordinates.* The lines of curvature on a surface in ordinary space are a metrically defined conjugate net; precisely, they are the only orthogonal conjugate net on the surface. Besides their special metric properties they, of course, also possess all the general projective properties of an arbitrary conjugate net. Moreover, the configurations associated with a point of an arbitrary conjugate net in the projective differential theory certainly are defined for the lines of curvature and may have interesting metric properties. The investigation of this situation is facilitated by employing a transformation of coördinates, which it is the purpose of this section to produce.

More in detail, the transformation with which we are concerned here connects the local homogeneous coordinates of the projective theory with the

local non-homogeneous cartesian coördinates of the metric theory. The projective homogeneous coordinates are based on the tetrahedron whose vertices are the points x, ρ, σ, y, in the notation of Section 3, with suitably chosen unit point. The cartesian coördinates are those described in Section 4.

In order to prepare for the calculation of the equations of the transformation we observe that elimination of X from the first three of equations (38) leads to a system of the form (1) whose coefficients are defined by the formulas

$$\begin{aligned} &a = 1/r = D/D'', \quad b = (\log E^{1/2})_u + E^{1/2}R_2/R_1\rho_2, \\ (46)\qquad &d = d' = 0, \quad c = -(1/r)[(\log G^{1/2})_v - G^{1/2}R_1/R_2\rho_1], \\ &b' = -G^{1/2}/\rho_1, \qquad c' = E^{1/2}/\rho_2. \end{aligned}$$

The ray-points ρ, σ of the lines of curvature are therefore given by the formulas

$$(47)\qquad \rho = x_u - c'x, \qquad \sigma = x_v - b'x,$$

while the point τ (hitherto denoted by y) which is the harmonic conjugate of the point x with respect to the foci of the axis of the lines of curvature is given * by two equal expressions,

$$(48)\quad \tau = (1/D)[x_{uu} - bx_u + Mx] = (1/D'')[x_{vv} + rcx_v + r(M + d)x]$$

in which M is defined by placing

$$(49)\qquad -2M = d + ab'^2 + c'^2 + ab_v' + c_u' + b'c - bc'.$$

In the first expression for τ let us substitute the value of x_{uu} taken from the first of equations (38), and then let us use the values of the coefficients of system (1) that are defined by the formulas (46). After a somewhat lengthy reduction whose details need not be recorded here, we find

$$(50)\quad \tau = (1/D)[-(E^{1/2}R_2/R_1\rho_2)x_u + (E/G^{1/2}\rho_1)x_v + DX + Mx],$$

where M is given by

$$(51)\quad -2M = D[(R_2 - R_1 + R_2\rho_{1v}/G^{1/2})/\rho_1^2 + (R_1 - R_2 - R_1\rho_{2u}/E^{1/2})/\rho_2^2].$$

To continue the process of expressing quantities in terms of R_1, R_2, ρ_1, ρ_2 and their derivatives, we may, if we like, use the formulas

$$(52)\quad E^{1/2} = \rho_2(1/R_2)_u/(1/R_1 - 1/R_2), \quad G^{1/2} = -\rho_1(1/R_1)_v/(1/R_2 - 1/R_1).$$

The actual calculation of the equations of the transformation proceeds

* Green, Second Memoir, p. 292.

as follows. Let $x_1, \cdots, x_4$ be the local projective homogeneous coordinates of a point P referred to the tetrahedron x, ρ, σ, τ at a point (x, y, z) of the surface under consideration, with the unit point chosen so that the general projective coördinates of the point P are represented by the expression

$$x_1 x + x_2 \rho + x_3 \sigma + x_4 \tau.$$

Let us replace ρ, σ, τ in this expression by their values from equations (47), (50) and then in place of (x, X) substitute in turn $(1, 0)$, (x, X), (y, Y), (z, Z). Let us divide the first expression thus obtained into each of the remaining three expressions. The resulting ratios are the general cartesian coördinates $\bar{x}, \bar{y}, \bar{z}$ of the point P. Denoting by ξ, η, ζ the local cartesian coördinates of P, we arrive by way of the calculations just indicated at a system of equations of precisely the same form as (44), with ξ, η, ζ defined by the formulas

$$\begin{aligned} \xi &= (E^{1/2} x_2 - R_2 x_4/\rho_2)/T, \\ \eta &= (G^{1/2} x_3 + R_1 x_4/\rho_1)/T, \qquad (53) \\ \zeta &= x_4 T, \end{aligned}$$

where the denominator T is defined by

$$T = x_1 - E^{1/2} x_2/\rho_2 + G^{1/2} x_3/\rho_1 + M x_4/D. \qquad (54)$$

These are the equations of the transformation which we were seeking. Solving them for the ratios of $x_1, \cdots, x_4$ we get the equations of the inverse transformation,

$$\begin{aligned} x_1 &= 1 + \xi/\rho_2 - \eta/\rho_1 + (\zeta/2)[(R_1 + R_2 \\ &\quad + R_2 \rho_{1v}/G^{1/2})/\rho_1^2 + (R_2 + R_1 - R_1 \rho_{2u}/E^{1/2})/\rho_2^2], \\ x_2 &= (\xi + R_2 \zeta/\rho_2)/E^{1/2}, \qquad (55) \\ x_3 &= (\eta - R_1 \zeta/\rho_1)/G^{1/2}, \\ x_4 &= \zeta. \end{aligned}$$

6. *Metric Results.* The formulas and the transformation discussed in the preceding sections will now be applied in studying for the lines of curvature some of the configurations associated with a general conjugate net in the projective theory. The usual invariants that occur in the projective theory will be calculated in terms of metric quantities for the lines of curvature, and a few metric theorems will be stated.

Since the equations of the osculating planes of the lines of curvature

C_u, C_v at a point P of a surface are $x_3=0$ and $x_2=0$ respectively in the homogeneous coördinate system, it follows immediately from equations (55) that the equations of the same planes in the cartesian system are

$$\rho_1\eta - R_1\zeta = 0, \qquad \rho_2\xi + R_2\zeta = 0. \tag{56}$$

The axis of the lines of curvature C_u, C_v at the point P is the line of intersection of these two planes and therefore passes through the point

$$(-R_2/\rho_2, \quad R_1/\rho_1, \quad 1).$$

The coördinates of the point τ on the axis result from multiplying each of these coördinates by D/M. The direction cosines of the axis can easily be written. The projective homogeneous local coördinates of the ray-points ρ, σ of the lines of curvature at the point P are $(0, 1, 0, 0)$, $0, 0, 1, 0)$, and hence the cartesian local coördinates of the ray-points are $(-\rho_2, 0, 0)$, $(0, \rho_1, 0)$. The distance between these points and the direction cosines of the ray can easily be written. *The cosine of the angle θ between the ray and axis of the lines of curvature* may thus be expressed by the formula

$$\cos\theta = (R_1 - R_2)/(\rho_1^2 + \rho_2^2)^{1/2}(1 + R_1^2/\rho_1^2 + R_2^2/\rho_2^2)^{1/2}. \tag{57}$$

The ray and axis are ordinarily not orthogonal.

The invariants appearing in equations (6) are given for the lines of curvature by the following formulas:

$$\begin{aligned}
&H = G^{1/2}(1/\rho_1)_u, & &K = -E^{1/2}(1/\rho_2)_v,\\
&\boldsymbol{H} = H + (\log R_1)_{uv}, & &\boldsymbol{K} = K + (\log R_2)_{uv},\\
&W^{(u)} = \boldsymbol{H} - K, & &W^{(v)} = \boldsymbol{K} - H,\\
&8\mathfrak{B}' = (\log R_1^3/R_2)_v, & &8\mathfrak{C}' = (\log R_2^3/R_1)_u,\\
&\mathfrak{D} = D[(R_2 - R_1 + R_2\rho_{1v}/G^{1/2})/\rho_1^2 + (R_2 - R_1 + R_1\rho_{2u}/E^{1/2})\rho_2^2], \\
&r = D''/D.
\end{aligned} \tag{58}$$

The projective theorem that *a curve C_u is a cone curve* (i. e. is the curve of contact of a cone circumscribing the surface) *in case $H=0$*, becomes the metric theorem that *the curve C_u is a cone curve in case it has constant geodesic curvature.* But the dual projective theorem that the curve C_u is a plane curve in case $\boldsymbol{H}=0$ does not seem to have so simple a metric formulation.

Since necessary and sufficient conditions that a net be quadratic are $\mathfrak{B}' = \mathfrak{C}' = 0$, it follows that *necessary and sufficient conditions that a surface be a quadric are*

$$(54) \qquad R_1 = cU^3/V, \qquad R_2 = cU/V^3,$$

where c is an arbitrary constant and U, V are arbitrary functions of u alone and of v alone respectively. The u-curves and v-curves form W congruences in case $W^{(u)} = W^{(v)} = 0$, and then the net is called an R net; it follows that *the lines of curvature on a surface form an R net if, and only if,*

$$(55) \qquad H - K + (\log R_1)_{uv} = 0, \qquad K - H + (\log R_2)_{uv} = 0.$$

In this case it follows that $(\log K_t)_{uv} = 0$.

Some calculation, which will be omitted, results in the formulas

$$(56) \qquad \begin{aligned} &(\log E/G)_{uv} = -2(H - K), \\ &(\log D/D'')_{uv} = (\log R_2/R_1)_{uv} - 2(H - K). \end{aligned}$$

Thus we arrive at the well-known theorem that the lines of curvature are isothermally orthogonal in case $H = K$, and also at the theorem that *the lines of curvature are isothermally conjugate if, and only if,*

$$(57) \qquad (\log R_2/R_1)_{uv} = 2(H - K).$$

It follows that *the lines of curvature are both isothermally orthogonal and isothermally conjugate if, and only if,*

$$(58) \qquad H = K, \qquad (\log R_2/R_1)_{uv} = 0.$$

In this case the lines of curvature are a conjugate net of the type called a *net of Jonas,* since they are isothermally conjugate and have equal point invariants H, K.

The equations of *the ray-point cubic of the pencil of conjugate nets determined by the lines of curvature* are easily found, by carrying out the transformation (55) on equations (17), to be

$$(59) \qquad \begin{aligned} \zeta = {} & (\xi^2/R_1 + \eta^2/R_2)[1 + (1/\rho_2 - \mathfrak{C}'/E^{1/2})\xi - (1/\rho_1 + \mathfrak{B}'/G^{1/2})\eta)] \\ & + \mathfrak{C}'\xi^3/R_1E^{1/2} - 3\mathfrak{B}'\xi^2\eta/R_1G^{1/2} - 3\mathfrak{C}'\xi\eta^2/R_2E^{1/2} + \mathfrak{B}'\eta^3/R_2G^{1/2} = 0. \end{aligned}$$

Similarly the equations of *the ray conic of the pencil determined by the lines of curvature* are found from equation (18) to be

$$(60) \qquad \begin{aligned} \zeta = {} & (\mathfrak{B}'^2 + r\mathfrak{C}'^2)(\xi^2/R_1 + \eta^2/R_2) \\ & - D''[1 + (1/\rho_2 - \mathfrak{C}'/E^{1/2})\xi - (1/\rho_1 + \mathfrak{B}'/G^{1/2})\eta]^2 = 0. \end{aligned}$$

The equations (20) of the quadrics of Darboux become, in the metric coördinate system,

$$(61) \qquad \xi^2/R_1 + \eta^2/R_2 + 2\zeta(-1 + k_2\xi + k_3\eta + k\zeta) = 0,$$

where k is arbitrary and k_2, k_3 are given by

$$(62) \qquad 4E^{1/2}k_2 = (\log K_t)_u, \qquad 4G^{1/2} = (\log K_t)_v.$$

From these formulas we can easily prove the well-known theorem that the line of centers of the quadrics of Darboux coincides with the normal at every point of a surface if, and only if, the surface has constant total curvature. The value of k for the quadric of Lie is expressed * by the formula

$$(63) \qquad \begin{aligned} 4k = {} & 2(k_2{}^2R_1 + k_3{}^2R_2) + (1/R_1 + 1/R_2)(k_2{}^2R_1{}^2 \sin^2 A \\ & + k_3{}^2R_2{}^2 \cos^2 A + 1) + (k_2R_1 \sin A)_u/E^{1/2} \sin A \\ & + (k_3R_2 \cos A)_v/G^{1/2} \cos A, \end{aligned}$$

where A is an angle such that

$$(64) \qquad (R_1 - R_2)\sin^2 A = -R_2, \qquad (R_1 - R_2)\cos^2 A = R_1.$$

We conclude with the remark that the conjugate osculating quadrics (26), (27) of Section 2, when defined as on the lines of curvature, are quite analogous to the asymptotic osculating quadrics as to their covariantive properties, the projectively covariant asymptotic curves in the latter instance being replaced by the metrically covariant lines of curvature in the former.

University of Chicago,
Chicago, Ill.

* Demoulin, "Sur quelques propriétés des surfaces courbes," *Comptes Rendus*, Vol. 147 (1908), p. 566.

ON THE EXPANSION OF HARMONIC FUNCTIONS IN TERMS OF NORMAL-ORTHOGONAL HARMONIC POLYNOMIALS.*

By Gaylord M. Merriman.†

Consider a closed, rectifiable Jordan curve, C, lying in the (x, y)-plane, possessing continuous curvature, and of length σ. Let $f(x, y)$ be a function which is harmonic inside C and continuous in the closed region $\bar{C}$ consisting of C and its interior. The present paper is concerned with the development of $f(x, y)$ in terms of a set of harmonic polynomials $P_m(x, y)$, which are linear combinations of the functions $1, x, y, x^2 - y^2, xy, \cdots$. Let

$$(1) \qquad f(x, y) \sim \sum_{j=0}^{\infty} c_j P_j(x, y)$$

be such an expansion, the c_j being constants obtained later in the Fourier fashion:

$$c_j = (1/\sigma) \int_C f(x, y) P_j(x, y)\, d\sigma, \qquad (j = 0, 1, 2, \cdots).$$

It is natural, of course, to desire that the n-th partial sum of the development:

$$s_n(x, y) = \sum_{j=0}^{n} c_j P_j(x, y)$$

should give among all polynomials $S_n(x, y)$ of the same degree a "best" approximation to $f(x, y)$, that is, say, that the integral

$$(2) \qquad (1/\sigma) \int_C [f(x, y) - S_n(x, y)]^2 d\sigma$$

should be minimized by the substitution $S_n(x, y) = s_n(x, y)$. It is well-known, however, that such a result can be accomplished by normalizing and orthogonalizing the polynomials $P_m(x, y)$: here it will be done with respect to the perimeter of C ‡:

---

* Presented to the American Mathematical Society, April, 1928.

† This paper was begun while the author was National Research Fellow in Mathematics, Harvard University, 1926-1928.

‡ St. Bergmann, *Mathematische Annalen*, Vol. 86 (1922), pp. 238-271, has normalized and orthogonalized polynomials with respect to an *area*; he considers in general expansions which can be obtained from real parts of analytic functions.

$$(3)\quad (1/\sigma)\int_C P_m(x,y)P_n(x,y)d\sigma = \left\{\begin{matrix} 0, & m \neq n \\ 1, & m = n \end{matrix}\right\}, \quad (m, n = 0, 1, \cdots).$$

We are first interested in exhibiting such a set of polynomials.

Next to discuss is the convergence of the set of normalized and orthogonalized polynomials; in this connection we shall prove the following theorem:

THEOREM I. *Let C be a closed, rectifiable Jordan curve possessing continuous curvature, and let $f(x,y)$ be harmonic interior to C and continuous in the closed region $\bar{C}$ consisting of C and its interior. Then the series* (1) *of harmonic polynomials normalized and orthogonalized with respect to the contour C converges uniformly to $f(x,y)$ in any closed region wholly interior to C.*

Finally, we prove the expansion (1) to be "overconvergent" in the sense of J. L. Walsh,* in that it converges uniformly in a larger region than that originally considered, thus extending the region of definition of $f(x,y)$.† In this connection we shall establish

THEOREM II. *Let C be a closed, rectifiable Jordan curve in the (x,y)-plane, and possessing continuous curvature, and let $w = \phi(z)$, $z = x + iy$, map the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve in the z-plane corresponding to $w = R$, $R > 1$. If $f(x,y)$ is harmonic interior to C and continuous in the closed region $\bar{C}$ consisting of C and its interior, then the series* (1) *of normal-orthogonal polynomials converges to $f(x,y)$ throughout the interior of C_R, and uniformly in any closed region wholly interior to C_R.*

The method of normalization and orthogonalization for harmonic polynomials has been used previously in an incomplete way by S. Bernstein,

---

* Two papers of Walsh, important in what follows, will henceforth be referred to as Walsh (i) and (ii): Walsh (i): "On the Overconvergence of Sequences of Polynomials of Best Approximation," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 794-815; Walsh (ii): "On the Degree of Approximation to a Harmonic Function," *Bulletin of the American Mathematical Society*, Sept.-Oct. 1927. The present reference is, of course, to the first of these, p. 794.

† Another point of view is, of course, to consider $f(x,y)$ defined outside C by harmonic extension.

Picone, and Brillouin.* Szégö † has used normalized and orthogonalized polynomials to expand analytic functions of a complex variable; in the paper just referred to, however, he considered only analytic Jordan curves as boundaries, a restriction which can be removed to some extent by a more recent theorem of Walsh,‡ so that some of his convergence theorems hold for the general rectifiable Jordan curve. In comparison, it is of interest that the present treatment of a similar problem for harmonic functions is, by the methods of proof, limited to rectifiable Jordan curves possessing continuous curvature.

It will be noted that the results of the paper furnish a solution of the Dirichlet problem for the region bounded by C and the boundary values $f(x,y)$ on C; the theoretical and practical value of such solutions of that problem has been remarked at the end of Walsh (i).

I. *The Set of Polynomials.* Let $r(z^m)$ be the real part, and $p(z^m)$ the coefficient of i, in z^m, $z=x+iy$, $i=(-1)^{1/2}$. Let $a_{j,k}$ be defined as follows:

$$a_{0,0}=1,$$

$$a_{0,2k-1}=(1/\sigma)\int_C p(z^k)\,d\sigma, \qquad a_{0,2k}=(1/\sigma)\int_C r(z^k)\,d\sigma, \quad (k=1,2,\cdots),$$

$$a_{2j-1,0}=(1/\sigma)\int_C p(z^j)\,d\sigma, \qquad a_{2j,0}=(1/\sigma)\int_C r(z^j)\,d\sigma, \quad (j=1,2,\cdots),$$

$$a_{2j-1,2k-1}=(1/\sigma)\int_C p(z^j)p(z^k)\,d\sigma, \qquad a_{2j-1,2k}=(1/\sigma)\int_C p(z^j)r(z^k)\,d\sigma, \quad (j,k\geqq 1),$$

$$a_{2j,2k-1}=(1/\sigma)\int_C r(z^j)p(z^k)\,d\sigma, \qquad a_{2j,2k}=(1/\sigma)\int_C r(z^j)r(z^k)\,d\sigma, \quad (j,k\geqq 1).$$

These numbers are the coefficients of symmetric, positive-definite quadratic bilinear forms in n variables, $n=1,2,3,\cdots$, since, for example,

---

* S. Bernstein, *Comptes Rendus*, Vol. 148, pp. 1306-1308; Picone, *Rendiconti dei Lincei* (1922), pp. 357-359; Brillouin, *Annales de Physique*, Vol. 6 (1916), pp. 137-223.

† G. Szégö, "Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören," *Mathematische Zeitschrift*, Vol. 9 (1921), pp. 218-270.

‡ Walsh, "Über die Entwicklung einer analytischen Funktion nach Polynomen," *Mathematische Annalen*, Vol. 96, pp. 430-436.

$$\sum_{j,k=0}^{2q} a_{j,k}t_jt_k=(1/\sigma)\int_C [t_0+t_1p(z)+t_2r(z)+\cdots+t_{2q}r(z^q)]^2d\sigma.$$

Also, the determinants D_n of these forms are positive for all n.*

The polynomials $P_m(x,y)$ are defined in the following manner:

$$P_0(x,y)=1,$$

$$P_{2m-1}(x,y)=\gamma_{2m-1}\begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{2m-1,0} \\ a_{0,1} & a_{1,1} & a_{2,1} & \cdots & a_{2m-1,1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-2} & a_{1,2m-2} & a_{2,2m-2} & \cdots & a_{2m-1,2m-2} \\ 1 & p(z) & r(z) & \cdots & p(z^m) \end{vmatrix},$$

$$P_{2m}(x,y)=\gamma_{2m}\begin{vmatrix} a_{0,0} & a_{1,0} & a_{2,0} & \cdots & a_{2m-1,0} & a_{2m,0} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & a_{2,2m-1} & \cdots & a_{2m-1,2m-1} & a_{2m,2m-1} \\ 1 & p(z) & r(z) & \cdots & p(z^m) & r(z^m) \end{vmatrix},$$

for $m=1,2,\cdots$, the γ's being constants to be determined later.

It is verified from the definitions that

(4a)
$$(1/\sigma)\int_C P_{2m}(x,y)r(z^k)\,d\sigma=\gamma_{2m}\begin{vmatrix} a_{0,0} & a_{1,0} & \cdots & a_{2m,0} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & \cdots & a_{2m,2m-1} \\ a_{0,2k} & a_{1,2k} & \cdots & a_{2m,2k} \end{vmatrix}$$

$$=\begin{Bmatrix} 0, & k<m \\ \gamma_{2m}D_{2m}, & k=m \end{Bmatrix},\qquad (m=1,2,\cdots).$$

Also,

(4b)
$$(1/\sigma)\int_C P_{2m}(x,y)p(z^k)\,d\sigma=\gamma_{2m}\begin{vmatrix} a_{0,0} & a_{1,0} & \cdots & a_{2m,0} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{0,2m-1} & a_{1,2m-1} & \cdots & a_{2m,2m-1} \\ a_{0,2k-1} & a_{1,2k-1} & \cdots & a_{2m,2k-1} \end{vmatrix}$$

$$=0,\quad k\leqq m,\quad m=1,2,\cdots.$$

---

* This is a well-known property of these forms; cf. Osgood's *Advanced Calculus*, p. 179.

Again,

(4c) $$(1/\sigma)\int_C P_{2m-1}(x,y)p(z^k)\,d\sigma = \left\{\begin{array}{ll} 0, & k < m \\ \gamma_{2m-1}D_{2m-1}, & k = m \end{array}\right\} \quad (m = 1, 2, \cdots),$$

and finally,

(4d) $$(1/\sigma)\int_C P_{2m-1}(x,y)r(z^k)\,d\sigma = 0, \qquad (k < m,\ m = 1, 2, \cdots).$$

Equations (4) insure the orthogonality of the polynomials as defined; the polynomials will now be proved normalized by a proper choice of the γ's.

Thus, if the polynomials are to be normalized,

$$1 = (1/\sigma)\int_C P^2_{2m}(x,y)\,d\sigma$$
$$= \gamma_{2m}D_{2m-1}\cdot(1/\sigma)\int_C P_{2m}(x,y)r(z^m)\,d\sigma = \gamma^2_{2m}D_{2m-1}D_{2m},$$

whence

$$\gamma_{2m} = 1/(D_{2m-1}D_{2m})^{1/2},$$

the positive square root being in order if we assume that the leading coefficient of $P_{2m}(x,y)$ is positive. Similarly,

$$\gamma_{2m-1} = 1/(D_{2m-2}D_{2m-1})^{1/2}.$$

With these values of γ_{2m} and γ_{2m-1} inserted in their definitions the polynomials $P_m(x,y)$, $m = 0, 1, 2, \cdots$, are both normalized and orthogonalized with respect to C.

II. *Convergence of the Set Interior to C.* The function $f(x,y)$, harmonic in the interior of C and continuous in $\bar{C}$, is now considered to be developed formally as in (1) and inquiry is made as to the convergence of the series. The clue is furnished by a discussion of the minimum value of the integral (2) with

$$S_n(x,y) = s_{2n}(x,y) = \sum_{j=0}^{2n} c_jP_j(x,y).$$

Here,

(2′) $$M_n(\{c_j\}) = (1/\sigma)\int_C [f(x,y) - s_{2n}(x,y)]^2\,d\sigma \geqq 0.$$

The minimum of $M_n(\{c_j\})$, call it m_n, is actually attained, and in obtaining it in the usual way it is found that

(5) $$c_j = (1/\sigma)\int_C f(x,y)P_j(x,y)\,d\sigma, \qquad (j = 0, 1, 2, \cdots),$$

in the Fourier fashion. In these circumstances the minimum value is

$$m_n = (1/\sigma) \int_C f^2(x, y)\, d\sigma - \sum_{j=0}^{2n} c_j^2. \tag{6}$$

It is next to be shown that m_n has the limiting value zero as n becomes infinite. In the first place, since $f(x, y)$ is harmonic in the interior of C and continuous in the closed region $\bar{C}$, it follows that, for a suitably chosen polynomial $P(x, y)$, it can be uniformly approximated in C *:

$$| f(x, y) - P(x, y) | < \epsilon, \tag{7}$$

where ϵ has been previously assigned. Hence,

$$(1/\sigma) \int_C [f(x, y) - P(x, y)]^2 d\sigma < \epsilon^2. \tag{8}$$

But consider

$$\lim_{n\to\infty} (1/\sigma) \int_C [f(x, y) - s_{2n}(x, y)]^2 d\sigma.$$

It is a monotonically decreasing function, and positive or zero. If, however, its limit is positive, say greater than ϵ^2, then $P(x, y)$ would, according to (8), give a better approximation to $f(x, y)$ than does $s_{2n}(x, y)$; this is impossible, so that the limit is zero. Hence, making use of (5) and (6),

$$(1/\sigma) \int_C f^2(x, y)\, d\sigma = \sum_{j=0}^{\infty} c_j^2,$$

and we have established

LEMMA 1. *If $f(x, y)$, expanded formally as in* (1) *with coefficients* (5), *is harmonic in the interior of C and continuous in the closed region $\bar{C}$ consisting of C and its interior, then*

$$\lim_{n\to\infty} (1/\sigma) \int_C [f(x, y) - s_{2n}(x, y)]^2 d\sigma = 0,$$

and

$$(1/\sigma) \int_C f^2(x, y)\, d\sigma = \sum_{j=0}^{\infty} c_j^2.$$

---

* This result has been proved for a general Jordan curve by Walsh: *Crelle's Journal*, Vol. 159 (1928), pp. 197-209, Satz II. It is this result which permits the extension of Szégö's work already alluded to.

COROLLARY. *If* $\bar{f}(x, y)$ *satisfies the conditions imposed on* $f(x, y)$ *in Lemma* 1, *and if the coefficients of its formal expansion in terms of the* $P_m(x, y)$ *are* $\bar{c}_j$, $j = 0, 1, 2, \cdots$, *then the* $\bar{c}_j$ *are not all zero unless* $\bar{f}(x, y) \equiv 0$.

LEMMA 2. *If* $f_n(x, y)$ *is harmonic in the region bounded by* C *and is continuous in* $\bar{C}$, *and if*

$$\lim_{n\to\infty} \int_C f_n^2(x, y)\, d\sigma = 0,$$

then

$$\lim_{n\to\infty} f_n(x, y) = 0$$

uniformly in any closed region wholly interior to C.

Green's integral gives, for any point (x, y) inside C,

$$f_n(x, y) = (1/2\pi) \int_C f_n(\xi, \eta) [\partial G(\xi, \eta)/\partial N]\, d\sigma,$$

where (ξ, η) is a general boundary point of C, N is the normal at (ξ, η), and $G(x, y)$ is the Green's function belonging to the region. Because of the condition of continuous curvature of C, $\partial G/\partial N$ is continuous on C and less in absolute value than some constant M. By use of Schwarz's inequality,

$$(9) \qquad | f_n(x, y) | \leqq [M'(\sigma)^{1/2}/2\pi] \left[\int_C f_n^2(\xi, \eta)\, d\sigma \right]^{1/2}.$$

The lemma follows at once from (9).*

Theorem I is an immediate consequence of Lemmas 1 and 2, if the substitution

$$f_n(x, y) = [f(x, y) - s_{2n}(x, y)]/\sigma^{1/2}$$

is made in Lemma 2.

---

* It will be noticed that all the results up to that of Lemma 2 can be proved for an arbitrary, rectifiable Jordan curve; for such a curve, however, lack of information concerning Green's function and its normal derivative (if it has one) preclude the use of Green's integral as above. We have therefore assumed the curve C as one having continuous curvature, in which case the normal derivative of the Green's function is continuous; this condition could, it is interesting to note, be lightened if conditions on $f(x, y)$ were correspondingly strengthened: cf. O. D. Kellogg, "Potential Functions" and "Double Distributions and the Dirichlet Problem," *Transactions of the American Mathematical Society*, Vol. 9 (1908), pp. 39 and 51, with cross references to previous results, especially those of Painlevé and Neumann.

III. *Overconvergence of the Set.* In connection with Theorem VII, p. 808, of Walsh (i) there is specific mention made of the result contained in Theorem II of the present paper (not then published), as being a previously established special case of the aforesaid Theorem VII; however, the asymptotic formula for $P_m(x, y)$ outside C, from which Theorem II was then deduced, has since collapsed. Inasmuch as Theorem VII of Walsh (i) indicates the truth of Theorem II, we include its statement merely for the sake of completeness of the present paper as first conceived. Walsh does not display a proof of Theorem VII, stating it as an analogue of Theorem III of the same paper, dealing with polynomials of a best approximation to a function of a complex variable; suffice it to say here that the methods of proof of the last mentioned theorem, in conjunction with Theorem I of Walsh (ii), a mapping theorem of Carathéodory,* and the use of Green's integral, establish Theorem II.

HARVARD UNIVERSITY,
THE UNIVERSITY OF CINCINNATI.

* *Mathematische Annalen* Vol. 72 (1912), pp. 126-127.

ON SOME PROBLEMS OF TCHEBYCHEFF.

By J. Geronimus.

We show in this paper that some results of Tchebycheff can be generalized and obtained without the use of algebraic continuous fractions. We base our investigation on some well known properties of orthogonal polynomials the theory of which can be made independent of continuous fractions.

1. *On functions having the least deviation from zero.* We, first, consider, with Tchebycheff, the following problem:

The polynomial of the n-th degree

$$y(x)=\sum_{i=0}^{n}\sigma_i x^{n-i} \tag{1}$$

is monotonic in the interval $(-1,+1)$. *Find the least deviation from zero of this polynomial in the interval* $(-1,+1)$ *its first coefficient* σ_0 *being given.**

Without loss of generality we may suppose that $y(-1)=0$. Then our polynomial may be written

$$y(x)=\int_{-1}^{x}\phi(x)\,dx, \tag{2}$$

where $\phi(x)\geq 0$ for $-1\leq x\leq 1$. We suppose that our monotonic polynomial is increasing in the interval $(-1,+1)$. It is easy to show further that $\phi(x)$ *is of the form*

$$\phi(x)=(1-x)^{\alpha}(1+x)^{\beta}u^2(x), \tag{3}$$

where $u(x)$ *is a polynomial, and* $\alpha=\beta=0$ *or* $\alpha=1,\ \beta=1$ *if n is odd, and* $\alpha=0,\ \beta=1$ *or* $\alpha=1,\ \beta=0$ *if n is even.*

Thus we are to minimize the integral

$$y(1)=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}u^2(x)\,dx. \tag{4}$$

In view of the well known minimum properties of orthogonal Tchebycheff

---

* P. Tchebycheff, "Sur les fonctions qui diffèrent le moins possible de zéro," *Oeuvres*, t. II (1907), pp. 189-215.

polynomials, we make use of the normalized Jacobi polynomials $P_k(x)$, corresponding to the interval $(-1, +1)$ with the characteristic function

$$p(x)=(1-x)^\alpha(1+x)^\beta.$$

We find, putting $u(x)=a_mP_m(x)=a_m(d_mx^m+\cdots)$:

$$\min y(1)=a_m{}^2,$$

where $n\sigma_0=(-1)^\alpha a_m{}^2d_m{}^2 \qquad (2m+\alpha+\beta=n-1)$.

since $y'(x)=n\sigma_0x^{n-1}+\cdots=(1-x)^\alpha(1+x)^\beta u^2(x)$.

Hence

$$(5)\qquad y(x)=\frac{n\sigma_0(-1)^\alpha}{d_m{}^2}\int_{-1}^{x}(1-x)^\alpha(1+x)^\beta P_m{}^2(x)\,dx,$$

$$(6)\qquad y(1)=\frac{2^{2m+\alpha+\beta+1}\,|\,\sigma_0\,|\,m!\,(m+\alpha+\beta)!\,(m+\alpha)!\,(m+\beta)!}{\{(2m+\alpha+\beta)!\}^2}.\text{*}$$

For large values of n we find, using Stirling's formula,

$$(7)\qquad y(1)\sim\frac{\pi\,|\,\sigma_0\,|\,n}{2^{n-1}}.$$

This problem of Tchebycheff may be generalized as follows:

Find the minimal deviation from zero in the interval $(-1, +1)$ *of a polynomial*

$$y(x)=\sum_{i=0}^{n}\sigma_ix^{n-i}$$

which is monotonic of order $h+1$ *in this interval, its coefficient* σ_l *being given* $(0\leq l\leq n)$.

A polynomial is said to be monotonic of order $h+1$ on a given interval if its $h+1$ first derivatives do not change sign in this interval.† Supposing that $n\to\infty$, while l and h are finite we have ‡

$$(8)\qquad \begin{aligned} y(1)&\sim\frac{\pi\,|\,\sigma_{2k}\,|\,n^{h-k+1}k!}{2^{n-2k-1}h!}, &&(l=2k),\\ y(1)&\sim\frac{\pi\,|\,\sigma_{2k+1}\,|\,n^{h-k+1}k!}{2^{n-2k-2}h!\,[h+(h^2+1)^{1/2}]}, &&(l=2k+1).\end{aligned}$$

---

* Pòlya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. II, Ss. 292-293.

† S. Bernstein, "Sur les polynomes multiplement monotones qui s'écartent le moins de zéro," *Comptes Rendus*, t. 185 (1927), p. 247.

‡ J. Geronimus, "Sur le polynome multiplement monotone qui s'écarte le moins de zéro dont un coefficient est donné," *Bulletin de l'Académie des Sciences de l'URSS, Classe des Sciences Physico-Mathématiques*, N 4 (1929), pp. 388-389.

For $h=0$, i. e. for ordinary monotonic polynomials, these two formulae may be combined into one:

$$(9) \qquad y(1) \sim \frac{\pi \mid \sigma_l \mid \nu!}{2^{n-l-1} \cdot n^{\nu-1}}, \qquad \nu = [l/2],$$

($[m]$ denotes the integral part of m).

The problem of Tchebycheff may be generalized in another way:

Find the least deviation from zero in the interval $(-1, +1)$ *of a polynomial*

$$y(x) = \sigma_0 x^n + \sigma_1 x^{n-1} + \cdots + \sigma_l x^{n-l} + \cdots + \sigma_n,$$

which is monotonic of order $h+1$ *in this interval, its first* $l+1$ *coefficients* $\sigma_0, \sigma_1, \cdots, \sigma_l$ *being given.*

The writer solved this problem for $l=1$, i. e. supposing that σ_0 and σ_1 are given. Assuming that $n \to \infty$, while h is finite, we have *

$$(10) \qquad \begin{aligned} y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{h+1}}{2^{n-1} h!} \{1 + (\sigma_1/\sigma_0 - h)^2\}, && \mid \sigma_1/\sigma_0 - h \mid \leq 1, \\ y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{h+1}}{2^{n-2} h!} \cdot \mid \sigma_1/\sigma_0 - h \mid, && \mid \sigma_1/\sigma_0 - h \mid \geq 1. \end{aligned}$$

For the particular case $h=0$, S. Bernstein gives the solution of this problem for all finite values of l, supposing that the given coefficients $\sigma_0, \sigma_1, \cdots, \sigma_l$ are of the same order of magnitude: †

$$y(1) \sim \frac{\pi \mid \sigma_0 \mid n^{l/2+1}}{2^{n-1} (l/2)!} \qquad (l \text{ even}),$$

$$(11) \qquad \left. \begin{aligned} y(1) &\sim \frac{\pi \mid \sigma_0 \mid n^{(l+1)/2}}{2^{n-1} [(l-1)/2]!} \{1 + (\sigma_1/\sigma_0)^2\}, && \mid \sigma_1 \mid \leq \mid \sigma_0 \mid \\ y(1) &\sim \frac{\pi \mid \sigma_1 \mid n^{(l+1)/2}}{2^{n-2} [(l-1)/2]!} && \mid \sigma_1 \mid \geq \mid \sigma_0 \mid \end{aligned} \right\} (l \text{ odd}).$$

We may also modify Tchebycheff's problem in the following manner:

Find the minimal deviation from zero in the interval $(-1, +1)$ *of a monotonic polynomial*

$$y(x) = \sum_{i=0}^{n} \sigma_i x^{n-i}$$

its coefficients $\sigma_{n-1}, \sigma_{n-2}, \cdots, \sigma_{n-k}$ *being given.*

---

* J. Geronimus, "Sur le polynome multiplement monotone, qui s'écarte le moins de zéro, dont les deux premiers coefficients sont donnés," *Comptes Rendus de l'Académie des Sciences de l'URSS* (1928), pp. 489-490.

† S Bernstein, "Zusatz zum vorangehenden Artikel der Herren W. Břečka und J. Geronimus," *Mathematische Annalen*, Bd. 102 (1929), S. 518.

Supposing that $n \to \infty$ while k is finite and assuming that all given coefficients are of the same order of magnitude, we obtain *

$$y(1) \sim \frac{2\pi\sigma_{n-1}}{n}\left[\frac{(\nu+1)!}{2^\nu[(\nu/2)!]^2}\right]^2, \tag{12}$$

where $\nu = k - 1$, if k is odd and $\nu = k - 2$, if k is even.

2. *On the ratio of two integrals taken between the same limits.*†

Find the extreme values of the ratio

$$\int_{-1}^{1} y(x)u(x)\,dx : \int_{-1}^{1} y(x)v(x)\,dx, \tag{13}$$

where $y(x)$ is a polynomial of degree $\leq n$; this polynomial and the function $v(x)$ are not negative for $-1 \leq x \leq 1$.

It is easy to show that $y(x)$ must be of the form †

$$\begin{aligned} &y(x) = (1-x)^\alpha(1+x)^\beta Z^2(x), \\ &Z(x) = \sum_{i=0}^{m-1} a_i x^i, \qquad (\alpha, \beta = 0, 1;\ \alpha + \beta + 2m - 2 = n). \end{aligned} \tag{14}$$

Putting, with Tchebycheff,

$$(1-x)^\alpha(1+x)^\beta u(x) = \theta_0(x), \quad (1-x)^\alpha(1+x)^\beta v(x) = \theta(x), \tag{15}$$

we must find the extreme values of the integral

$$\int_{-1}^{1} \theta_0(x) Z^2(x)\,dx,$$

being given the value of another integral

$$\int_{-1}^{1} \theta(x) Z^2(x)\,dx = 1, \qquad [\theta(x) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1].$$

Applying the classical method of Analysis, we find the conditions of extremum

$$\int_{-1}^{1} Z(x)[\theta_0(x) - \lambda\theta(x)]x^k\,dx = 0, \qquad (k = 0, 1, 2, \cdots, m-1). \tag{16}$$

Hence the new parameter λ represents the required extremum

$$\frac{\int_{-1}^{1} \theta_0(x) Z^2(x)\,dx}{\int_{-1}^{1} \theta(x) Z^2(x)\,dx} = \lambda. \tag{17}$$

* W. Břečka und J. Geronimus, "Ueber das monotone Polynom, welches die minimale Abweichung von Null hat, wenn die Werte seiner ersten Ableitungen gegeben sind," *Mathematische Annalen*, Bd. 102 (1929), S. 514.

† Tchebycheff, *Oeuvres*, T. II, pp. 377-402.

To derive an equation for λ, introduce the quantities

$$\int_{-1}^{1}[\theta_0(x)-\lambda\theta(x)]x^k dx = c_k, \qquad (k=0,1,2,\cdots), \tag{18}$$

and rewrite (16) in the following form:

$$\sum_{i=0}^{m-1} a_i c_{s+i} = 0, \qquad (s=0,1,2,\cdots,m-1). \tag{19}$$

It is clear that the determinant

$$\| c_{ik} \| = 0, \qquad (i,k=0,1,\cdots,m-1), \quad c_{ik}=c_{i+k} \tag{20}$$

vanishes, for otherwise we would have $a_0=a_1=\cdots=a_{m-1}=0$. Thus we see that

$$\lambda_2 \leq \frac{\int_{-1}^{1}\theta_0(x)Z^2(x)\,dx}{\int_{-1}^{1}\theta(x)Z^2(x)\,dx} \leq \lambda_1, \tag{21}$$

where λ_1 is the largest and λ_2 is the smallest root of the equation (20).

Particular case:

$$\theta_0(x) = (ax+b)^r\theta(x), \qquad (r \text{ positive integer}). \tag{22}$$

Here we can solve our problem without using the equation (20). The conditions (16) of extremum are

$$\int_{-1}^{1}Z(x)[(ax+b)^r-\lambda]\theta(x)x^k dx = 0, \quad (k=0,1,2,\cdots,m-1), \tag{23}$$

and show that

$$Z(x)[(ax+b)^r-\lambda] = \sum_{s=0}^{r-1} b_s\phi_{m+s}(x), \tag{24}$$

where the orthogonal and normal polynomials $\{\phi_i(x)\}$ correspond to the interval $(-1,+1)$ with the characteristic function $\theta(x)$. Hence

$$Z(x) = \frac{\sum_{s=0}^{r-1} b_s\phi_{m+s}(x)}{(ax+b)^r-\mu^r}, \qquad (\mu^r=\lambda). \tag{25}$$

Since $Z(x)$ is a polynomial, we have necessarily

$$\sum_{s=0}^{r-1} b_s\phi_{m+s}(\nu_k) = 0, \qquad (k=1,2,\cdots,r), \tag{26}$$

$$\nu_k = (\mu e^{2\pi ik/r}-b)/a.$$

Thus we get finally (since not all b_s are 0)

$$\mu_2{}^r \leqq \frac{\int_{-1}^{1} \theta(x)(ax+b)^r Z^2(x)\,dx}{\int_{-1}^{1} \theta(x) Z^2(x)\,dx} \leqq \mu_1{}^r \tag{27}$$

where μ_1 and μ_2 are resp. the largest and the smallest roots of the equation

$$\begin{vmatrix} \phi_m(\nu_1) & \phi_{m+1}(\nu_1) & \cdots & \phi_{m+r-1}(\nu_1) \\ \phi_m(\nu_2) & \phi_{m+1}(\nu_2) & \cdots & \phi_{m+r-1}(\nu_2) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \phi_m(\nu_r) & \phi_{m+1}(\nu_r) & \cdots & \phi_{m+r-1}(\nu_r) \end{vmatrix} = 0. \tag{28}$$

The case $\theta(x)=(1-x)^h$, $\theta_0(x)=(1-x)^{h+1}$ has been discussed by S. Bernstein, who found *

$$\mu_2 \sim 2u/m^2, \qquad (m \to \infty), \tag{29}$$

where u is the smallest root of Bessel's function

$$\mathfrak{J}_h(2u^{1/2})=(u^{1/2})^h \sum_{s=0}^{\infty} \frac{(-1)^r u^r}{r!\,(h+r)!}\ . \tag{30}$$

In case $\theta(x)=(1-x)^h$, $\theta_0(x)=(1-x)^{h+2}$ we find † that μ_1 is the largest and μ_2 is the smallest root of the equation

$$P_m(1-\mu)P_{m+1}(1+\mu)-P_{m+1}(1-\mu)P_m(1+\mu)=0, \tag{31}$$

$P_k(x)$ being the normalized Jacobi polynomial corresponding to the interval $(-1, +1)$ with the characteristic function $p(x)=(1-x)^h$. It follows, that

$$\mu_2 \sim 2z_0/m^2, \qquad (m \to \infty), \tag{32}$$

where z_0 is the smallest positive root of the integral function

$$F(z)=\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\,(h+k)!\,(h+2k+1)!}\ . \tag{33}$$

---

* S. Bernstein, "Sur les polynomes multiplement monotones," *Communications de la Société Mathématique de Kharkow*, IV série, t. I (1927), p. 9.

† W. Břečka und J. Geronimus, "Ueber die monotone Polynome, welche die minimale Abweichung von Null haben," *Mathematische Zeitschrift*, Bd. 30 (1929), Ss. 366-369.

3. *On extreme values of sums involving a polynomial and its derivatives.* The following problem has been solved by Tchebycheff *:

Find the extreme values of the sum

$$\sum_{i=1}^{n} F(x, y, y', y'', \cdots)_{x=x_i} \tag{34}$$

at certain given points $x_1, x_2, \cdots, x_n$, *where* F *is a given polynomial in* $x, y, y', \cdots$.

We shall consider a particular case of this problem:

Find the minimum of the sum

$$L = \sum_{i=1}^{n} \theta(x_i)[y(x_i) - f(x_i)]^2, \tag{35}$$

where $\theta(x_i) > 0$, $(i = 1, 2, \cdots, n)$, $f(x)$ *is a given polynomial of degree* $r \leq n - 1$ *and*

$$y(x) = \sum_{s=0}^{m} A_s x^s, \qquad m < r, \tag{36}$$

the coefficients $A_{i_1}, A_{i_2}, \cdots, A_{i_\nu}$ *being given.*

Here again we use the orthogonal and normal Tchebycheff polynomials determined by

$$\sum_{i=1}^{n} \theta(x_i)\phi_k(x_i)\phi_s(x_i) = \begin{cases} 0, & k \neq s, \\ 1, & k = s \leq n - 1. \end{cases} \tag{37}$$

{They necessarily exist as has been shown by Tchebycheff}. Putting

$$\begin{gathered} y(x) = \sum_{k=0}^{m} a_k \phi_k(x), \qquad f(x) = \sum_{k=0}^{r} b_k \phi_k(x), \\ y(x) - \sum_{k=0}^{m} b_k \phi_k(x) = \sum_{k=0}^{m} c_k \phi_k(x) = y_1(x), \end{gathered} \tag{38}$$

we see that we are to minimize the sum

$$L_1 = L - \sum_{k=m+1}^{r} b_k^2 = \sum_{k=0}^{m} c_k^2 \tag{39}$$

under conditions

$$\sum_{k=0}^{m} c_k \phi_k^{(i_s)}(0) = i_s!\, A_{i_s} - \sum_{k=0}^{m} b_k \phi_k^{(i_s)}(0) = a_s, \qquad (s = 1, 2, \cdots, \nu). \tag{40}$$

Using the classical method we obtain the conditions of extremum

* P. Tchebycheff, "Des maxima et minima des sommes composées des valeurs d'une fonction entière et de ses dérivées," *Oeuvres*, t. II, pp. 3-40; *Journal des Mathématiques pures et appliquées*, II série, t. XIII (1868), p. 9-42.

$$c_k = \sum_{s=1}^{\nu} \lambda_s \phi_k^{(i_s)}(0), \qquad (k = 0, 1, 2, \cdots, m), \tag{41}$$

whence

$$L_1 = \sum_{s=1}^{\nu} \lambda_s a_s. \tag{42}$$

Further we have

$$a_r = \sum_{s=1}^{\nu} \lambda_s a_{sr}, \qquad (r = 1, 2, \cdots, \nu), \tag{43}$$

where we have put

$$a_{kr} = \sum_{j=0}^{m} \phi_j^{(i_k)}(0)\, \phi_j^{(i_r)}(0), \qquad (k, r = 1, 2, \cdots, \nu). \tag{44}$$

From (42) and (43) we see that L may be found from the equation

$$\begin{vmatrix} L - \sum_{k=m+1}^{r} b_k^2 & a_1 & a_2 & \cdots & a_\nu \\ a_1 & a_{11} & a_{21} & \cdots & a_{\nu 1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_\nu & a_{1\nu} & a_{2\nu} & \cdots & a_{\nu\nu} \end{vmatrix} = 0. \tag{45}$$

The polynomial $y(x)$ for which this minimum is attained may be found from the equation

$$\begin{vmatrix} y(x) - \sum_{k=0}^{m} b_k \phi_k(x) & R_1(x) & R_2(x) & \cdots & R_\nu(x) \\ a_1 & a_{11} & a_{21} & \cdots & a_{\nu 1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_\nu & a_{1\nu} & a_{2\nu} & \cdots & a_{\nu\nu} \end{vmatrix} = 0, \tag{46}$$

where

$$R_s(x) = \sum_{r=0}^{m} \phi_r(x) \phi_r^{(i_s)}(0), \qquad (s = 1, 2, \cdots, \nu). \tag{47}$$

In particular, if $\nu = 1$ and the coefficient A_i is given then

$$L = \sum_{k=m+1}^{r} b_k^2 + \frac{\{i!\, A_i - \sum_{k=0}^{m} b_k \phi_k^{(i)}(0)\}^2}{\sum_{k=0}^{m} \{\phi_k^{(i)}(0)\}^2}, \tag{48}$$

and

$$y(x) = \sum_{k=0}^{m} b_k \phi_k(x) + \frac{i!\, A_i - \sum_{k=0}^{m} b_k \phi_k^{(i)}(0)}{\sum_{k=0}^{m} \{\phi_k^{(i)}(0)\}^2} \cdot \sum_{k=0}^{m} \phi_k^{(i)}(0) \phi_k(x). \tag{49}$$

KHARKOW (UKRAINA).

THREE NOTES ON CHARACTERISTIC EXPONENTS AND EQUATIONS OF VARIATION IN CELESTIAL MECHANICS.

By AUREL WINTNER.

I. Upon the Characteristic Exponents of the Celestial Mechanics.

II. Upon the Characteristic Exponents in the Strömgrenian Groups of Periodic Orbits.

III. Upon the Equation of Jacobi for Dynamical Systems with Two Degrees of Freedom.

Appendix. On a Theorem in the Pfaffian Dynamics of Birkhoff.

I. UPON THE CHARACTERISTIC EXPONENTS OF THE CELESTIAL MECHANICS.

In the following there is given a proof of a theorem, first enunciated by Poincaré,* a satisfactory proof for which is not to be found in the works of Poincaré nor in the later literature.† In particular, the theorems of Poincaré upon the position of the characteristic exponents of a dynamical system with two degrees of freedom, for instance in the case of the restricted problem of three bodies,‡ are proven, probably for the first time.

If $A(t) = \| a_{jk}(t) \|$ is a matrix of continuous functions which are defined for $-\infty < t < +\infty$, then the system of differential equations

$$\dot{x}_j = \sum_{k=1}^{N} a_{jk}(t) x_k \qquad (j = 1, 2, \cdots, N) \tag{1}$$

possesses in the interval $-\infty < t < +\infty$ one and only one solution $x_j = x_j(t)$ which satisfies the N initial conditions $(x_j(t))_{t=0} = x_j(0)$. The matrix $X(t) = \| x_{jk}(t) \|$ of N linearly independent solutions

$$x_1 = x_{m1}(t), \quad x_2 = x_{m2}(t), \cdots, \quad x_N = x_{mN}(t); \qquad (m = 1, 2, \cdots, N) \tag{2}$$

* H. Poincaré, *Méthodes nouvelles de la Mécanique Céleste*, Vol. 1, p. 192.

† Poincaré proceeds (*loc. cit.*) on the assumption that if $x_j = x_j(t)$ is a solution of the differential system (1) [cf. above] and the coefficients of the differential system possess the period T, there must exist a constant matrix $\| a_{jk} \|$ so that

$$x_j(t + T) = \sum_k a_{jk} x_k(t) \tag{*}$$

identically in t. It is clear that (*) is not valid [even if one interprets (*) symbolically and conceives of the $x_j(t)$ as vectors, namely as different solutions of (1)].

‡ *Loc. cit.*, Vol. 3, p. 343-344.

is called a *fundamental matrix* of (1). The determinant of the matrix $X(t)$ which satisfies the Jacobian identity

$$\det X(t) = \det X(0) \cdot \exp \int_0^t - \sum_{k=1}^{N} a_{kk}(\tau)\,d\tau, \tag{3}$$

obviously either vanishes identically in t, or vanishes for no value of t. The fundamental matrices are accordingly characterized by

$$\det X(0) \neq 0. \tag{4}$$

If $X(t)$ is a fundamental matrix, then a matrix $Y(t)$ is then and only then a fundamental matrix, if there exists a non-singular matrix K with constant elements, for which

$$Y(t) = KX(t). \tag{5}$$

The matrix K is obviously uniquely determined by the matrices $X(t)$ and $Y(t)$ (Principle of Superposition).

We shall place $N = 2n$ and shall assume the existence of a quadratic form H in the variables x_j (the coefficients of which are given functions of t) of such a character that the system (1) can be written in the canonical form

$$\dot{x}_{2i-1} = \partial H/\partial x_{2i}, \quad \dot{x}_{2i} = -\partial H/\partial x_{2i-1} \qquad (i = 1, 2, \cdots, n = N/2). \tag{6}$$

In order that this be possible, it is necessary and sufficient that the $4n^2$ elements $a_{jk}(t)$ of the *coefficient matrix* $A(t)$ satisfy the following $2n^2 - n$ conditions arising from the peculiar symmetry of the Hamiltonian equations:

$$\begin{aligned} a_{2i-1\,2h} &= a_{2h-1\,2i}, \\ a_{2i\,2h-1} &= a_{2h\,2i-1}, \\ a_{2i-1\,2h-1} &= -a_{2h\,2i}, \end{aligned} \qquad (h, i = 1, 2, \cdots, n). \tag{7}$$

We note in particular that if the matrix $A(t)$ satisfies the above conditions for Hamiltonian symmetry we have $a_{2i2i} = -a_{2i-1\,2i-1}$, so that the sum under the integral sign in (3) vanishes identically and $\det X(t)$ is independent of t. In addition if $x_j = x^{(1)}{}_j(t)$, $x_j = x^{(2)}{}_j(t)$ are two arbitrary solutions of (1) the determinant sum

$$c = \sum_{i=1}^{n} \begin{vmatrix} x^{(1)}{}_{2i-1}(t) & x^{(1)}{}_{2i}(t) \\ x^{(2)}{}_{2i-1}(t) & x^{(2)}{}_{2i}(t) \end{vmatrix} \tag{8}$$

is, as was proven by Poincaré,* independent of t. In order to demonstrate this it is only necessary to calculate the time derivative $\dot{c}$ of (8) which, as follows from (1) and (7), is identically zero.

* *Loc. cit.*, Vol. 1, p. 166-167.

It is accordingly possible to introduce, for a given fundamental matrix $X(t) = \| x_{jk}(t) \|$, by means of the definition

$$C_X = \| c_{jk} \| ; \qquad (j, k = 1, 2, \cdots, 2n-1, 2n), \tag{8}$$

$$c_{jk} = \sum_{i=1}^{n} \begin{vmatrix} x_{j\,2i-1}(t) & x_{j\,2i}(t) \\ x_{k\,2i-1}(t) & x_{k\,2i}(t) \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} x_{j\,2i-1}(0) & x_{j\,2i}(0) \\ x_{k\,2i-1}(0) & x_{k\,2i}(0) \end{vmatrix}$$

a constant matrix of $4n^2$ elements. This matrix is skew-symmetric (but not necessarily real) and will be called the *commutator matrix* of the fundamental matrix $X(t)$. It follows readily from (8) that we have

$$\det C_X = [\det X(0)]^2. \tag{9}$$

The elements of the commutator matrix of the fundamental matrix KX in (5) may be calculated from (8) and one obtains for the general element of C_{KX}

$$\sum_{p=1}^{2n} \sum_{q=1}^{2n} l_{jp} c_{pq} l_{kq} \tag{10}$$

in which we understand $K = \| l_{jk} \|$.

We shall now suppose that the matrix $A(t)$ of the coefficients of (1) is periodic and not a constant matrix; hence it possesses a primitive period T such that

$$a_{jk}(t+T) = a_{jk}(t) ; \qquad (j, k = 1, 2, \cdots, N). \tag{11}$$

It follows from (11) that the matrix $X(t+T)$ is, simultaneously with $X(t)$, a matrix of solutions of (1) and, from the superposition principle, that there exists for every fundamental matrix $X(t)$ one and only one constant matrix Γ_X for which

$$X(t+T) = \Gamma_X X(t). \tag{12}$$

This matrix Γ_X will be called the *characteristic matrix* of $X(t)$. It is clear, from (8) and (12), that we have

$$C_{LX} = C_X \text{ where } L = \Gamma_X, \tag{13}$$

and for the fundamental matrix (5) by means of (12), that

$$KX(t+T) = K\Gamma_X X(t) = K\Gamma_X K^{-1} \cdot KX(t). \tag{14}$$

One can obviously combine (10) and (14) into the following pair of formulae:

$$C_{KX} = KC_X K', \qquad \Gamma_{KX} = K\Gamma_X K^{-1}. \tag{15}$$

In these equations the accent is used to denote the transposed matrix and

K is any non-singular, constant matrix so that (15), as follows from the superposition principle (5), exhibits the connection between the various commutator and characteristic matrices of the different fundamental matrices of (1).

It follows from (4), (9) and (12) that for every fundamental matrix $X(t)$ we have

(16) $$\det C_X \neq 0, \qquad \det \Gamma_X \neq 0.$$

From the second formula of (15) it is clear that the equation of the $2n$-th degree

(17) $$\det(\lambda E - \Gamma_X) = 0,$$

together with the elementary divisors, is invariant if one introduces a new fundamental matrix $X(t)$, so that one may speak simply of the invariants (i. e. characteristic constants and elementary divisors) of the characteristic group belonging to the periodic coefficient matrix $A(t)$. From the first equation of (15) we obtain because of (13)

(18) $$C_X = \Gamma_X C_X \Gamma'_X.$$

Since C_X and Γ_X are non-singular [cf. (16)] we may write this equation in the form

(19) $$\Gamma'_X = C^{-1}{}_X \Gamma^{-1}{}_X C_X.$$

Consequently Γ'_X and $\Gamma^{-1}{}_X$ and therefore Γ_X and $\Gamma^{-1}{}_X$ possess the same characteristic constants. Now the characteristic constants of the reciprocal matrix of a matrix M are always the reciprocals of the characteristic constants of the matrix M. *We therefore conclude that* (17) *is a reciprocal equation,* i. e., by a suitable choice of the notation for the characteristic constants of the characteristic group we may write

(20) $$\lambda_{2i-1}\lambda_{2i} = 1, \qquad (i = 1, 2, \cdots, n).$$

We may infer that if the coefficient matrix $A(t)$ is real, the coefficients of the algebraic equation (17) are evidently real and therefore complex characteristic numbers λ_j of the characteristic group can occur only in conjugate pairs.

If now we introduce in place of the "multipliers" λ_j the "characteristic exponents" ρ_j defined by the equations

(21) $$\rho_j = -(-1)^{1/2}(T/2\pi)\log\lambda_j; \qquad (j = 1, 2, \cdots, 2n-1, 2n),$$

(which are only determined mod 1), equations (20) are equivalent to

(22) $$\rho_{2i-1} + \rho_{2i} = 0; \qquad (i = 1, 2, \cdots, n).$$

It is desirable to emphasize at this point that the characteristic group becomes illusory if (11) is fulfilled for every T, or in other words, if the coefficient matrix A of (1) is a constant matrix (this case has previously been excluded). If T is arbitrary, the characteristic matrix Γ_X of a given fundamental matrix $X(t)$ is in no case determined by (12) and indeed, because T in (12) is arbitrary, a continuum of different determinations is possible. It would be accordingly useless to seek for a direct connection, by means of continuity considerations between the characteristic constants of a constant matrix A, and the characteristic constants of the characteristic group which are only defined for the non-constant, periodic coefficient matrices (11). Such attempts in the literature, for example in the case of the small periodic orbits about the Lagrangian libration points of the restricted problem of three bodies, have led to various misunderstandings. On the contrary certain continuity considerations of Liapounoff * can readily be justified in the following manner: For a coefficient matrix A, independent of t, we introduce in the system (1) a new independent variable t defined by †

$$\tau = -(-1)^{1/2} \log t. \tag{21'}$$

The Hamiltonian character of the differential equations obviously remains unchanged and the coefficient matrix of the new differential system, provided that A is not the zero matrix, is a periodic function of t possessing the primitive period 2π. One perceives quite readily that the invariants of the characteristic group of the new differential system coincide, up to the conformal transformation (21), with the invariants of the original differential system from which it follows that the characteristic constants ω_j of a constant matrix A satisfying the conditions (7) are connected not by (20) but by the relation ‡

* A. Liapounoff, *Annales de Toulouse* (2), 1907, p. 413. The continuity considerations of Liapounoff (in the application of the method of Liapounoff) are not permissible for a periodic solution of a dynamical problem since in this case there always occurs a multiple root in the characteristic equations. This arises from (20) and the well known fact that for the equations of variations belonging to a periodic solution (which is not independent of t) at least one root of the characteristic equation is unity.

† The transformation (21′) is nothing more than the transformation of Euler employed to transform a differential system (1) with constant coefficient matrix A (which is therefore not of the Fuchsian type) into a differential system of the Fuchsian type.

‡ The theorem (22′) has been proven (under an unessential restriction) by Birkhoff in a direct manner by an application of the method of Liapounoff. Cf. G. D. Birkhoff, "Dynamical Systems," *American Mathematical Society Colloquium Publications*, Vol. 9, pp. 77-78.

$$(22') \qquad \omega_{2i-1} + \omega_{2i} = 0; \qquad (i = 1, 2, \cdots, n).$$

We proceed now to the canonical equations of motion

$$(23) \qquad \dot{y}_{2i-1} = \partial F/\partial y_{2i}, \qquad \dot{y}_{2i} = -\partial F/\partial y_{2i-1}; \qquad (i = 1, 2, \cdots, n)$$

in which we shall assume that F is independent of t and possesses continuous second derivatives with respect to the arguments. We consider a one-parametric sheaf of curves

$$(24) \qquad y_j = \bar{y}_j(t;\epsilon); \qquad (j = 1, 2, \cdots, 2n-1, 2n)$$

possessing continuous second derivatives with respect to the variables t and ϵ, the member of the sheaf corresponding to $\epsilon = 0$ being a solution

$$(25) \qquad y_j = y_j^0(t) \equiv \bar{y}_j(t; 0)$$

of (23), the sheaf being otherwise perfectly arbitrary.

If we introduce the notation

$$\delta = (\partial/\partial\epsilon)_{\epsilon=0}, \qquad \delta^2 = \tfrac{1}{2}(\partial^2/\partial\epsilon^2)_{\epsilon=0}$$

in connection with the sheaf (24), the equations

$$(26) \qquad \delta\dot{y}_{2i-1} = \delta(\partial F/\partial y_{2i}), \qquad \delta\dot{y}_{2i} = -\delta(\partial F/\partial y_{2i-1})$$

are called the variational equations of (23) belonging to the solution (25). If we now write

$$(27) \qquad x_j = x_j(t) = \delta y_j \equiv (\partial\bar{y}_j(t;\epsilon)/\partial\epsilon)_{\epsilon=0}$$

and

$$(28) \qquad \delta^2 F = \sum_{j=1}^{2n} \sum_{k=1}^{2n} F^0_{y_j y_k}(t) x_j x_k,$$

where we understand

$$(29) \qquad F^0_{y_j y_k}(t) = F_{y_j y_k}(y_1^0(t), \cdots, y^0_{2n}(t)), \qquad F_{y_j y_k} = \partial^2 F/\partial y_j \partial y_k$$

and if we identify H with (28), equation (26) can be written in the form of (6). The variations (27) accordingly satisfy a differential system (1), the coefficients of which, as follows from the second part of (29), satisfy the conditions (7), and are uniquely determined by (29) and the initial solution (25). The coefficients of (1) are independent of the arbitrary sheaf (24) and if (25) is a periodic solution, it follows from (29) that (11) is also fulfilled.

II. Upon the Characteristic Exponents in the Strömgrenian Groups of Periodic Orbits.

In this note the theory of the characteristic exponents * is applied to the so-called groups † of periodic orbits. The final purpose is the treatment of a question which Strömgren and Burrau called to my attention, some time ago. Strömgren had found, by means of mechanical quadratures and harmonic analysis, in the Copenhagen group n of periodic solutions of the restricted problem of three bodies, an orbit ‡ which, while it is a simple orbit (i. e. a curve of Jordan), is nevertheless the limiting position of double orbits of the group each of which possesses two circuits before re-entering into itself. Burrau and Strömgren inquired if it would not be possible to find an analytical condition for such a coalescence of two circuits. This problem is simply the converse of the one treated by Poincaré § in his theory of the second "genre." The answer to the problem of Strömgren and Burrau obtained by the method of the present note has led to the conjecture that the results of Poincaré ¶ concerning the existence of his periodic solutions of the second "genre" cannot be correct without some additional restrictions. It has been easy to find an example, mentioned at the end of this note, showing that the existence statements of Poincaré ‖ (for which no satisfactory proof was given) are not valid even for the periodic groups of the restricted problem of three bodies.

We shall denote by

$$(1) \qquad \xi = \xi^0(t), \qquad \eta = \eta^0(t)$$

a given solution of the differential equations

$$(2) \qquad \ddot{\xi} - 2\dot{\eta} = \Omega_\xi(\xi, \eta), \qquad \ddot{\eta} + 2\dot{\xi} = \Omega_\eta(\xi, \eta); \qquad (\Omega_\zeta = \partial\Omega/\partial\zeta)$$

of the restricted problem of three bodies and by

$$(3) \qquad \xi = \bar{\xi}(t, \epsilon), \quad \eta = \bar{\eta}(t, \epsilon); \qquad \bar{\xi}(t, 0) = \xi^0(t), \quad \bar{\eta}(t, 0) = \eta^0(t)$$

any sheaf of curves (fulfilling the usual differentiability conditions) which

---

* H. Poincaré, *Les méthodes nouvelles de la Mécanique Céleste*, Vol. 3 (1899), Chap. XXVIII, and Chap. XXXI.

† E. Strömgren, "Forms of Periodic Motion in the Restricted Problem etc.," *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, Nr. 39 (1922).

‡ Cf., for instance, E. Strömgren, *loc. cit.*, p. 25-26.

§ Poincaré, *loc. cit.* *, p. 201 etc.

¶ Poincaré, *loc. cit.* *, p. 226 etc. or p. 331 etc.

‖ H. Poincaré, *loc. cit.* ¶, previous reference and also p. 351 or pp. 355-356.

for the value $\epsilon = 0$ of the parameter is identical with (1), without necessarily being a solution of (2) for $\epsilon \neq 0$. If one places, for any given function G or F of ξ and η,

$$\delta F^0(t) = [(\partial/\partial\epsilon)F(\bar{\xi}(t,\epsilon),\bar{\eta}(t,\epsilon))]_{\epsilon=0}, \qquad G^0(t) = G(\xi^0(t),\eta^0(t))$$

and employs the abbreviations $\Omega_{\xi\eta} = \partial^2\Omega/\partial\xi\partial\eta, \cdots$ and

$$(4) \qquad \begin{aligned} u = u(t) &= [(\partial/\partial\epsilon)\bar{\xi}(t,\epsilon)]_{\epsilon=0} = \delta\xi^0(t), \\ v = v(t) &= [(\partial/\partial\epsilon)\bar{\eta}(t,\epsilon)]_{\epsilon=0} = \delta\eta^0(t), \end{aligned}$$

the so-called variational equations of (2), belonging to (1), are

$$(5) \qquad \ddot{u} - 2\dot{v} = \Omega^0_{\xi\xi}(t)u + \Omega^0_{\xi\eta}(t)v, \quad \ddot{v} - 2\dot{u} = \Omega^0_{\eta\xi}(t)u + \Omega^0_{\eta\eta}(t)v.$$

The coefficients of these linear differential equations, belonging to the given solution (1) of (2), are independent of the special choice of the sheaf (3) so that any sheaf yields by means of (4) a solution of (5). The coefficients of the linear differential equations (5) have obviously the period T if the given solution (1) of (2) has the period T, as will be supposed in the following. We shall exclude the trivial case where (1) is one of the five equilibrium solutions of Lagrange, that is we shall suppose that (1) is not independent of t, so that

$$(6) \qquad u = \dot{\xi}^0(t), \qquad v = \dot{\eta}^0(t),$$

which obviously * constitute a solution of (5), are not identically zero. According to the classical theory of homogeneous linear differential equations with periodic coefficients there exists a uniquely determined real quartic equation

$$(7) \qquad \det(\lambda E - \Gamma) = 0, \quad \text{where} \quad \det\Gamma \neq 0,$$

the roots λ of which are characterized by the fact that (5) possesses at least one not identically vanishing solution with the multiplicative property

$$(8) \qquad u(t+T) = \lambda u(t), \qquad v(t+T) = \lambda v(t).$$

The condition (8) can be written in the form

$$(9) \qquad \begin{aligned} u(t) &= \phi(t)\exp 2\pi i\rho t/T, \qquad & v(t) &= \psi(t)\exp 2\pi i\rho t/T, \\ \phi(t+T) &= \phi(t), & \psi(t+T) &= \psi(t), \end{aligned}$$

where

$$(10) \qquad \lambda = \exp 2\pi i\rho/T.$$

---

* If we define the sheaf (3) with the use of (1) as $\bar{\xi}(t,\epsilon) = \xi^0(t+\epsilon)$, $\bar{\eta}(t,\epsilon) = \eta^0(t+\epsilon)$ the Jacobian rule (4) yields for (5) the solution (6).

If the four roots λ_j of the quartic equation (7) are distinct or at least if the corresponding elementary divisors are simple, equations (8) yield four linearly independent solutions of (5) and therefore also the general solution of (5). If not all elementary divisors are simple, there can not exist four linearly independent solutions with the multiplicative property (9) and the general solution will contain secular terms.

Since (1) has the period T and is not independent of t it follows from (6) and (8) that at least one of the four roots λ_j of the characteristic equation (7) must be equal to one, say

$$\lambda_1 = 1. \tag{11}$$

Furthermore it follows from (6) that not all coefficients of the linear differential equations (5) can be constant; otherwise the periodic solution (6) of (5) and therefore also the functions (1) would have the form

$$a + b\cos 2\pi(t - t_0)/T; \qquad (b \neq 0),$$

whereas the non-linear differential equations (2) of the restricted problem of three bodies do not possess a solution (1) of this elementary character. Since not all coefficients of (5) are independent of t it follows from a general theorem * on dynamical systems that the quartic equation (7) is a reciprocal one, that is we have $\lambda_1\lambda_2 = 1$, $\lambda_3\lambda_4 = 1$, or, according to (11),

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = \lambda^0, \quad \lambda_4 = 1/\lambda^0. \tag{12}$$

λ^0 is here a number which is not zero † and which can be real or complex; if it is complex it must be of modulus unity inasmuch as the complex roots of the real quartic equation can occur only in conjugate pairs. The number λ^0 which lies either on the real axis or on the boundary of the unit circle determines ‡ (in the sense of the characteristic exponents) the stability character of the given solution (1) or (2); cf. (9), (10), (12). The two § possible domains $\lambda^0 > 0$, $\lambda^0 < 0$ of instability are joined with the domain $|\lambda^0| = 1$ of stability at the two indifferent points $\lambda^0 = 1$, $\lambda^0 = -1$ respectively.¶ The indifferent case $\lambda^0 = -1$ is of special importance in what follows. The proofs are also valid for a periodic ejection solution (1)

---

* Cf. p. 608.

† Otherwise (7) would yield $\det \Gamma = 0$ whereas Γ is, according to (7), a non-singular matrix.

‡ The stability number λ^0 of (1) may be calculated, according to the method of Hill, from a linear differential equation of second order with periodic coefficients; cf. p. 617.

§ The two domains of instability have no common point; cf. †.

¶ This theorem has been stated, without a satisfactory proof, by Poincaré, *loc. cit.*, p. 343-344.

inasmuch as the above mentioned theorems hold also in the regularizing variables of Thiele.

We now suppose that we have instead of one periodic solution (1) an analytic sheaf *

$$(13) \qquad \xi = \xi^0(t; C), \qquad \eta = \eta^0(t; C)$$

of solutions of (2) with the period $T = T(C)$ so that (1) is contained in (13) for a special value of C. The sheaf (12) is, in the sense of Strömgren,† a "group" of periodic solutions (the mass ratio μ having an arbitrarily *fixed* value). The number λ^0 (or the corresponding characteristic exponent ρ^0) which defines the stability character ‡ of the path (1) is also a function of the parameter C of the group:

$$(13') \qquad \lambda^0 = \lambda^0(C) = \exp 2\pi i\, \rho^0(C)/T(C) \qquad \text{[cf. (10), (12)]}.$$

In order to apply the stability function (13′) of the group (13) to the problem of multiple paths mentioned in the introduction, we consider in the domain of the group parameter C a point $C = C_0$ having the property that the *primitive* (that is smallest) period of the solution (13) is for $C < C_0$ equal to $T(C)$ but for $C = C_0$ equal to $T(C_0)/p$ where p is a positive integer different from 1. In other words the path is for $C < C_0$ a "p-fold" orbit which shrinks for $C = C_0$ to a "simple" orbit, the primitive period being p-times smaller than $T(C)$ if $C = C_0$ (the integer p is in the case mentioned in the introduction equal to 2). We shall call $C = C_0$ a shrinking orbit of the order p. If we write

$$(14) \qquad T_0 = T(C_0), \qquad \lambda_0 = \lambda^0(C_0)$$

the primitive period of (13) for $C = C_0$ is T_0/p and

$$(15) \qquad \xi = \xi_0(t) = \xi^0(t/p; C_0), \qquad \eta = \eta_0(t) = \eta^0(t/p; C_0)$$

is a solution of (2) with the primitive period T_0. If we identify (1) with

* For the dynamical meaning of the parameter C of the sheaf cf. G. Herglotz, *Seeliger-Festschrift* (1924), pp. 197-199.

† Cf., for instance, *loc. cit.*

‡ It is necessary to refer the stability character of (13) to the characteristic exponent λ^0 alone, independent of the question whether (5) possesses a secular solution or not. Without this convention no group for which the period is not independent of the group parameter C would possess a stable domain. In order to show this it is only necessary to define the sheaf (3) with the use of (13) as $\bar{\xi}(t, \epsilon) = \xi^0(t; C + \epsilon)$, $\bar{\eta}(t, \epsilon) = \eta^0(t; C + \epsilon)$. The rule (4) yields in the case, if C is not a stationary point of the function $T = T(C)$, a secular solution of (5) [one need only differentiate the Fourier series of (13) term by term]. This secular solution, together with the periodic solution (6), furnishes two solutions of (5) belonging to $\lambda_1 = \lambda_2 = 1$.

(15) the rule (6) yields for (5) the particular solution

$$u=\dot{\xi}_0(t), \qquad v=\dot{\eta}_0(t) \tag{16}$$

where

$$\dot{\xi}_0(t+T_0)=\dot{\xi}_0(t), \qquad \dot{\eta}_0(t+T_0)=\dot{\eta}_0(t) \tag{17}$$

and T_0 is the *primitive* period of (16). If

$$u=u_j(t), \qquad v=v_j(t)\,; \qquad (j=1,2,3,4) \tag{18}$$

is any system of linear independent solutions of (5) the solution (16) may be represented in the form

$$\dot{\xi}_0(t)=\sum_{j=1}^{4} c_j u_j(t), \qquad \dot{\eta}_0(t)=\sum_{j=1}^{4} c_j v_j(t). \tag{19}$$

It has been pointed out that for the multiplicative condition (8) one can choose (18) in such a manner, that for a *fixed* value of j the solution $u=u_j(t)$, $v=v_j(t)$ either fulfills the two conditions

$$u_j(t+T_0/p)=\lambda_j u_j(t), \qquad v_j(t+T_0/p)=\lambda_j v_j(t) \tag{20}$$

or it contains a secular term. If $u=u_j(t)$, $v=v_j(t)$ contains a secular term the corresponding coefficient c_j must vanish inasmuch as the superposition (19) of (18) is periodic. It therefore follows from (20) that

$$c_j u_j(t+T_0/p)=c_j\lambda_j u_j(t), \quad c_j v_j(t+T_0/p)=c_j\lambda_j v_j(t) \tag{21}$$

holds for all four values of j whereas (20) was valid only for such values of j for which $u=u_j(t)$, $v=v_j(t)$ do not contain secular terms. From (12) and (13′) follows

$$\lambda_1=1, \quad \lambda_2=1, \quad \lambda_3=\lambda_0, \quad \lambda_4=1/\lambda_0 \tag{22}$$

and the equations (21) yield by p-fold iteration

$$c_j u_j(t+T_0)=c_j\lambda_j^p u_j(t), \quad c_j v_j(t+T_0)=c_j\lambda_j^p v_j(t). \tag{23}$$

Since the four solutions (18) are linearly independent we obtain from (17), (19) and (23) the four conditions $c_j(\lambda_j^p-1)=0$ which can be written, according to (22), in the form

$$c_3(\lambda_0^p-1)=0, \qquad c_4(\lambda_0^p-1)=0. \tag{24}$$

It is easy to see that it is not possible for c_3 and c_4 to be both zero and consequently (24) can be written in the form $\lambda_0^p-1=0$, that is λ_0 is a p-th root of unity. From $c_3=c_4=0$ it would follow, according to (19), (21) and (22), that (16) possesses the period T_0/p which is a contradiction,

the *primitive* period of (16) being, according to (15), equal to $T_0 > T_0/p$. From the assumption that λ_0 is not a *primitive* p-th root of unity one obtains in the same manner the same contradiction. We therefore have the following theorem:

In order that $C = C_0$ should be a shrinking point of the order p, it is necessary that the stability function (13′) of the group should be for $C = C_0$ a p-th primitive root of unity. In particular for the case $p = 2$ mentioned in the introduction we have the necessary condition

$$\lambda^0(C_0) = -1. \tag{25}$$

It may be pointed out that the p-fold iteration of the general solution of the variational equations used above is essentially the same as the topological iteration, initiated by Poincaré and Levi-Civita, and developed by Birkhoff in his researches on the corresponding surface transformations associated with dynamical systems of two degrees of freedom.

The question now arises whether the necessary condition just given is also a sufficient one; for instance, in the case $p = 2$, whether all paths $C = C_0$, for which the stability function (13′) fulfills the condition (25), must be simple limiting positions of double paths of the given group. The answer would be an affirmative one if the existence statements of Poincaré concerning his second "genre" were correct. However the conclusions of Poincaré are not satisfactory inasmuch as the main difficulty of the question, namely the explicit compatibility and reality discussion of certain finite non-linear equations of condition ("Verzweigungsgleichungen") is not treated in all its details. It it possible to give an example which shows that also the final result is wrong, i. e. that the second "genre" need not exist under the condition stated by Poincaré and that the necessary condition (25) is not a sufficient one.

If one treats in particular the group g of Strömgren, which originates with the moon orbits, in an analytical manner it is easy to show that the orbits are, for a sufficiently small *fixed* value of the mass ratio μ, simple orbits (curves of Jordan), insofar as the orbit does not lie too close to the Hecuba gap. In addition, if the mass ratio is sufficiently small, the group represents, exclusive of the immediate neighborhood of the Hecuba gap, a regular sheaf (without branch points with respect to the Jacobian constant C). Nevertheless there exist in the domain between the moon orbits and the Hecuba gap, namely in the vicinity of the Hestia commensurability, two values of C for which $\lambda^0(C) = -1$. This follows readily by a combination of previous results of Birkhoff, von Zeipel, and of the writer.—For details cf. a paper which will be published in the *Mathematische Zeitschrift.*

III. Upon the Equation of Jacobi for Dynamical Systems with Two Degrees of Freedom.

In this note the method of reduction of the equations of variation for a dynamical system of two degrees of freedom which Hill * employed in his treatment of the motion of the lunar perigree (in which he initiated the use of infinite determinants), is placed upon an analytical basis. In particular several unsatisfactory points in Poincaré's † treatment of the equation of normal displacement (which will be calculated explicitly) are removed.—There is given a proof of the isoenergetic equivalence of the reduced equation of the *second* order (equation of Jacobi) and the original equations of the *fourth* order, and in the course of the proof it will become clear how such a paradoxical fact is possible. For the special case where the variations are those of a periodic solution, one obtains the corresponding equivalence proof for the characteristic exponents and it is then easy to see precisely why the principle of Maupertuis ‡ can be employed to determine the pair of non-trivial characteristic exponents (in so far as the periodic solution is not an equilibrium solution).

The differential equations of the most general conservative dynamical system with two degrees of freedom can be reduced to the form §

$$(1)\qquad \ddot{x}-2\lambda(x,y)\dot{y}=\Omega_x(x,y),\quad \ddot{y}+2\lambda(x,y)\dot{x}=\Omega_y(x,y),\qquad (\cdot = d/dt),$$

where $\lambda=\lambda(x,y)$ and $\Omega=\Omega(x,y)$ are given functions of x and y. Let

$$(2)\qquad x=x^0(t),\qquad y=y^0(t)$$

be a given solution of (1). We shall use the abbreviations

* G. W. Hill, *Collected Mathematical Works,* Vol. 1 (1905), p. 244 ff. Cf. also G. H. Darwin, *Scientific Papers,* Vol. 4 (1911), p. 27 ff.

† H. Poincaré, *Méthodes nouvelles de la Mécanique Céleste,* Vol. 3 (1899), Chap. XXIX. In this respect cf. also a note of mine to appear in the volume for 1930 of the *Berichte de mathematisch-physischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig,* in which reference is made to the literature concerning the relation of the dynamical equation of Jacobi to the calculus of variations.

‡ Cf. H. Poincaré, *loc. cit.*, and G. D. Birkhoff, "The Restricted Problem of Three Bodies," *Rendiconti del Circolo Matematico di Palermo,* Vol. 39 (1915), § 3. In this paper of Birkhoff and in the one referred to in footnote §, the equation of Jacobi is obtained in a very simpel manner by an analytic continuation of the given solution in the complex domain of the independent variable, the arc length.

§ G. D. Birkhoff, "Dynamical Systems with Two Degrees of Freedom," *Transactions of the American Mathematical Society,* Vol. 18 (1917), p. 202 ff.

$$(3)\quad \begin{aligned} &F^0(t)=F(x^0(t),y^0(t)), \qquad F_x{}^0(t)=F_x(x^0(t),\,y^0(t)),\\ &F^0{}_{xy}(t)=F_{xy}(x^0(t),y^0(t)),\cdots;\ F_x=\partial F/\partial x,\ F_{xy}=\partial^2F/\partial x\partial y,\cdots, \end{aligned}$$

$F=F(x,y)$ being either the function $\lambda=\lambda(x,y)$ or the function $\Omega=\Omega(x,y)$.

A pair of functions

$$(4)\qquad u=u(t), \qquad v=v(t)$$

is called a variation belonging to the solution (2) if it satisfies the homogeneous differential equations

$$(5)\quad \begin{aligned} \ddot u-2\lambda^0(t)\dot v&=[\Omega^0{}_{xx}(t)+2\lambda_x{}^0(t)\dot y^0(t)]u+[\Omega^0{}_{xy}(t)+2\lambda_y{}^0(t)\dot y^0(t)]v,\\ \ddot v+2\lambda^0(t)\dot u&=[\Omega^0{}_{yx}(t)-2\lambda_x{}^0(t)\dot x^0(t)]u+[\Omega^0{}_{yy}(t)-2\lambda_y{}^0(t)\dot x^0(t)]v. \end{aligned}$$

The coefficients of this linear differential system of the fourth order are given functions of t, which are defined by (2) and (3). The differential system (5) possesses the homogeneous linear integral

$$(6)\qquad \dot x^0(t)\dot u+\dot y^0(t)\dot v-\Omega_x{}^0(t)u-\Omega_y{}^0(t)v=c,$$

the time derivative of (6) being, by means of (5), identically zero. The differential equations (5) are necessary and sufficient in order that

$$(4')\qquad x=x^0+u, \qquad y=y^0+v \qquad [\text{cf. } (2)]$$

should be, up to terms of second order in u, v, a solution of (1). We have, therefore, $\mu=\delta x$, $v=\delta y$ and (6) can be derived in a *formal* manner from the vis viva integral

$$(7)\qquad \tfrac{1}{2}(\dot x^2+\dot y^2)-\Omega(x,y)=C$$

of (1), if one places $c=\delta C$. We shall call, therefore, a variation (4), i. e. a solution of (5) then and only then an isoenergetic or Maupertuisian variation if the integration constant c in (6), determined by the four initial values of the solution (4) of (5), vanishes so that we have, for all values of t,

$$(8)\qquad \dot x^0(t)\dot u(t)+\dot y^0(t)\dot v(t)-\Omega_x{}^0(t)u(t)-\Omega_y{}^0(t)v(t)=0.$$

If one introduces (2) in (1) and in (7) and differentiates (1) and (7) with respect to t, one obtains exactly the relations (5) and (8), where

$$(9)\qquad u=\dot x^0(t), \qquad v=\dot y^0(t),$$

i. e. (9) is an isoenergetic variation of (2). We shall now suppose that the given initial solution (2) of (1) is not an equilibrium solution, i. e. the solution (9) of (5) is not identically zero. It is then possible to choose the origin $t=0$ of the t-axis in such a manner that for the given solution (2) of (1)

$$x^0(0) \neq 0, \quad y^0(0) \neq 0, \quad \dot{x}^0(t) \neq 0, \quad \dot{y}^0(0) \neq 0 \tag{10}$$

(it is of course possible that the solution (2) lies on a straight line of the (x, y)-plane, but one can rotate the coördinate system in such a manner that this straight line should not be parallel to either of the two coördinate axes). The solution (4) of (5), defined by the four initial values

$$u(0), \quad v(0), \quad \dot{u}(0), \quad \dot{v}(0), \tag{11}$$

is then and only then an isoenergetic variation if

$$\dot{x}^0(0)\dot{u}(0) + \dot{y}^0(0)\dot{v}(0) - \Omega_x{}^0(0)u(0) - \Omega_y{}^0(0)v(0) = 0, \tag{8'}$$

inasmuch as (6) is an integral of (5). It follows from (10) and (8′) that the manifold of the isoenergetic variations depends on three of the four arbitrary constants (11).

The projection of the variation $u = \delta x$, $v = \delta y$ on the orientated normal of the curve (2), belonging to a fixed value of t, is

$$\zeta(t) = \frac{-\dot{y}^0(t)}{\sqrt{\dot{x}^0(t)^2 + \dot{y}^0(t)^2}} u(t) + \frac{\dot{x}^0(t)}{\sqrt{\dot{x}^0(t)^2 + \dot{y}^0(t)^2}} v(t). \tag{12}$$

We put

$$\vartheta = \vartheta(t) = \begin{vmatrix} \dot{x}^0(t) & \dot{y}^0(t) \\ u(t) & v(t) \end{vmatrix} \tag{13}$$

so that

$$\vartheta = [(\dot{x}^0)^2 + (\dot{y}^0)^2]^{1/2} \zeta. \tag{14}$$

We shall call a given function of t then and only then an isoenergetic normal displacement of (2) if there exists at least one isoenergetic variation (4) by means of which the given function may be represented in the form (12). We want to show that the isoenergetic normal displacements of (2) can be characterised as the solutions of a linear differential equation of the second order (equation of Jacobi). First of all we notice the Lagrangian relation

$$\begin{aligned} &[X_2V_0 - Y_2U_0 + X_1V_1 - Y_1U_1][X_1X_2 + Y_1Y_2] \\ &- [U_1X_1 + V_1Y_1 - U_0X_2 - V_0Y_2][X_1Y_2 - X_2Y_1] \\ &+ [U_1Y_2 - V_1X_2][X_1{}^2 + Y_1{}^2] - [V_0X_1 - U_0Y_1][X_2{}^2 + Y_2{}^2] = 0 \end{aligned}$$

(used also by Poincaré) which is identically fulfilled for the ten independent variables. In particular we have

$$\begin{aligned} &\vartheta[\dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\ddot{y}^0(t)] \\ &- [\dot{u}\dot{x}^0(t) + \dot{v}\dot{y}^0(t) - u\ddot{x}^0(t) - v\ddot{y}^0(t)][\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)] \\ &+ [\dot{u}\dot{y}^0(t) - \dot{v}\dot{x}^0(t)][\dot{x}^0(t)^2 + \dot{y}^0(t)^2] - \vartheta[\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] = 0 \end{aligned} \tag{15}$$

if we use the abbreviated notation (13).

Let (4) be an isoenergetic variation so that (5) and (8) are fulfilled. From (1), (2), (3) follows

$$(16)\qquad \Omega_x{}^0(t) = \ddot{x}^0(t) - 2\lambda^0(t)\dot{y}^0(t), \quad \Omega_y{}^0(t) = \ddot{y}^0(t) + 2\lambda^0(t)\dot{x}^0(t)$$

and therefore from (8)

$$\dot{x}^0(t)\dot{u} + \dot{y}^0(t)\dot{v} - [\ddot{x}^0(t) - 2\lambda^0(t)\dot{y}^0(t)]u - [\ddot{y}^0(t) + 2\lambda^0(t)\dot{x}^0(t)]v = 0$$

i. e. by means of (13)

$$(17)\qquad -2\lambda_0(t)\vartheta = \ddot{x}^0(t)u + \ddot{y}^0(t)v - \dot{x}^0(t)\dot{u} - \dot{y}^0(t)\dot{v}.$$

The definition (13) of ϑ yields

$$(18)\qquad \ddot{\vartheta} + 2[\dot{u}\ddot{y}^0(t) - \dot{v}\ddot{x}^0(t)] = v\dddot{x}^0(t) - u\dddot{y}^0(t) + \ddot{v}\dot{x}^0(t) - \ddot{u}\dot{y}^0(t)$$

and from (16) and (4) follows

$$\begin{aligned}
\dddot{x}^0(t) &= 2\lambda^0(t)\ddot{y}^0(t) + [\Omega^0{}_{xx}(t) + 2\lambda_x{}^0(t)\dot{y}^0(t)]\dot{x}^0(t) \\
&\qquad + [\Omega^0{}_{xy}(t) + 2\lambda_y{}^0(t)\dot{y}^0(t)]\dot{y}^0(t), \\
\dddot{y}^0(t) &= -2\lambda^0(t)\ddot{x}^0(t) + [\Omega^0{}_{yx}(t) - 2\lambda_x{}^0(t)\dot{x}^0(t)]\dot{x}^0(t) \\
&\qquad + [\Omega^0{}_{yy}(t) - 2\lambda_y{}^0(t)\dot{x}^0(t)]\,\dot{y}^0(t), \\
\ddot{v} &= -2\lambda^0(t)\dot{u} + [\Omega^0{}_{yx}(t) - 2\lambda_x{}^0(t)\dot{x}^0(t)]u \\
&\qquad + [\Omega^0{}_{yy}(t) - 2\lambda_y{}^0(t)\dot{x}^0(t)]v, \\
\ddot{u} &= 2\lambda^0(t)\dot{v} + [\Omega^0{}_{xx}(t) + 2\lambda_x{}^0(t)\dot{y}^0(t)]u \\
&\qquad + [\Omega^0{}_{xy}(t) + 2\lambda_y{}^0(t)\dot{y}^0(t)]v.
\end{aligned}$$

If one multiplies these four equations by v, $-u$, $\dot{x}^0(t)$, $-\dot{y}^0(t)$ respectively and adds the products, one obtains

$$\begin{aligned}
&v\dddot{x}^0(t) - u\dddot{y}^0(t) + \ddot{v}\dot{x}^0(t) - \ddot{u}\dot{y}^0(t) \\
&= 2\lambda_0(t)[\ddot{x}^0(t)u + \ddot{y}^0(t)v - \dot{x}^0(t)\dot{u} - \dot{y}^0(t)\dot{v}] \\
&\quad + [\Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t) - 2\{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}][\dot{x}^0(t)v - \dot{y}^0(t)u],
\end{aligned}$$

i. e. by means of (18), (17) and (13)

$$(19)\qquad \ddot{\vartheta} + 2[\dot{u}\ddot{y}^0(t) - \dot{v}\ddot{x}^0(t)] = L(t)\vartheta,$$

where

$$(20)\quad L(t) = -4\lambda^0(t)^2 + \Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t) - \{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}.$$

Introducing (16) and (19) in (15) one obtains

$$\begin{aligned}
&\dot{\vartheta}[\dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\ddot{y}^0(t)] + 2\lambda^0(t)[\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)]\vartheta \\
&\quad + \tfrac{1}{2}[-\ddot{\vartheta} + L(t)][\dot{x}^0(t)^2 + \dot{y}^0(t)^2] - \vartheta[\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] = 0,
\end{aligned}$$

i. e.

$$\alpha(t)\ddot{\vartheta} + \beta(t)\dot{\vartheta} + \gamma(t)\vartheta = 0, \tag{21}$$

where [cf. (20)]

$$\begin{aligned} \alpha(t) &= -\tfrac{1}{2}[\dot{x}^0(t)^2 + \dot{y}^0(t)^2], \\ \beta(t) &= \dot{x}^0(t)\ddot{x}^0(t) + \dot{y}^0(t)\dot{y}^0(t), \\ \gamma(t) &= 2\lambda^0(t)[\dot{x}^0(t)\ddot{y}^0(t) - \dot{y}^0(t)\ddot{x}^0(t)] - [\ddot{x}^0(t)^2 + \ddot{y}^0(t)^2] \\ &\quad + \tfrac{1}{2}[\dot{x}^0(t)^2 + \dot{y}^0(t)^2][-4\lambda^0(t)^2 + \Omega^0_{xx}(t) + \Omega^0_{yy}(t) \\ &\quad - \{\dot{x}^0(t)\lambda_y{}^0(t) - \dot{y}^0(t)\lambda_x{}^0(t)\}]. \end{aligned} \tag{22}$$

If one places (14) in (21) there follows finally the self-adjoint equation

$$\ddot{\zeta} + \nu(t)\zeta = 0, \tag{23}$$

where

$$\nu(t) \tag{24}$$

is a given function of t, uniquely determined by (22), i. e. any isoenergetic normal dispalcement (2) is a solution of the differential equation (23). We shall now show that any solution of the differential equation (23) is an isoenergetic normal displacement of (2), in other words that there exists for any solution $\vartheta = \vartheta(t)$ of (21) at least one pair (4) of functions $u(t)$, $v(t)$ so that for the three functions $\vartheta(t)$, $u(t)$, $v(t)$ the conditions (5), (8) and (13) are fulfilled.

The arbitrarily given solution $\vartheta = \vartheta(t)$ of (21) is characterized by its initial values

$$\vartheta(0) \quad \text{and} \quad \dot{\vartheta}(0). \tag{25}$$

We determine four initial values (11) in the following manner. One of these, for instance $u(0)$, shall be chosen arbitrarily. The three other numbers, namely $v(0)$, $\dot{u}(0)$, $\dot{v}(0)$, shall fulfill the three conditions

$$\begin{aligned} &(26,1) \quad 0 = [-\vartheta(0) - \dot{y}^0(0)u(0)] + \dot{x}^0(0)v(0), \\ &(26,2) \quad 0 = [-\dot{\vartheta}(0) - \ddot{y}(0)u(0)] + \ddot{x}^0(0)v(0) - \dot{y}^0(0)\dot{u}(0) + \dot{x}^0(0)\dot{v}(0), \\ &(26,3) \quad 0 = [0 - \Omega_x{}^0(0)u(0)] - \Omega_y{}^0(0)v(0) + \dot{x}^0(0)\dot{u}(0) + \dot{y}^0(0)\dot{v}(0), \end{aligned}$$

which always determine the three constants uniquely, the determinant being $-\dot{x}^0(0)[\dot{x}^0(0)^2 + \dot{y}^0(0)^2] \neq 0$ [cf. (10)]. Let

$$u = u^*(t), \qquad v = v^*(t) \tag{27}$$

denote the solution of (5) which has the initial values (11). The integration constant c [cf. (6)] for the solution (27) is by means of (26.3) equal to zero, i. e. (26) is an isoenergetic variation and therefore

$$\vartheta^*(t) = \begin{vmatrix} \dot{x}^0(t) & \dot{y}^0(t) \\ u^*(t) & v^*(t) \end{vmatrix} \tag{28}$$

a solution of (21). Furthermore it follows from (26, 1), (26, 2) and (28)

$$\vartheta(0) = \vartheta^*(0), \qquad \dot{\vartheta}(0) = \dot{\vartheta}^*(0) \tag{25'}$$

i. e. the given solution $\vartheta(t)$ of (21) has the same initial values as the solution (28) and therefore $\vartheta(t) \equiv \vartheta^*(t)$ so that the arbitrarily given solution $\vartheta(t)$ may be represented with the use of an isoenergetic variation (4). This completes the demonstration of the theorem that a function $f(t)$ of t is then and only then an isoenergetic normal displacement of (2) if $\zeta = f(t)$ fulfills the differential equation (23).

The differential equation (23) is of the second order whereas there exists a three parametric series of isoenergetic variations (4) [cf. p. 619]. It follows that the isoenergetic normal representation (12) of a given solution $\zeta(t)$ of (23) is not uniquely determined inasmuch as it contains one arbitrary parameter [cf. also p. 621, where it was possible to choose one of the four integration constants (11) in an arbitrary manner]. The reason of this circumstance is obviously the following one. (9) and therefore also

$$u = a\dot{x}^0(t), \qquad v = a\dot{y}^0(t) \qquad (a = \text{constant} \neq 0) \tag{29}$$

is a solution of (5) and (8) which does not vanish identically [cf. p. 618] whereas the solution of (21) defined by (13) and (29) vanishes identically for any value of the integration constant a. We infer that the equation (21) or (23) is illusory if the given solution (2) of (1) is independent of t. It is of course allowed that both functions (9) should simultaneously vanish for certain isolated values of t but the limit values $\zeta(t+0)$, $\zeta(t-0)$ of (14) will exist also for these values of t.

If the function $\lambda(x, y)$ introduced by the Coriolis forces is identically zero the three functions $\lambda^0(t)$, $\lambda_x{}^0(t)$, $\lambda_y{}^0(t)$ will be zero for all values of t and (20) becomes

$$L(t) = \Omega^0{}_{xx}(t) + \Omega^0{}_{yy}(t). \tag{20'}$$

However this is not the case in the restricted problem of three bodies. For this problem is $\lambda(x, y) \equiv 1$ if x and y denote Cartesian coördinates and t denotes the time, so that one must add to (20′) the term

$$-4. \tag{20''}$$

If one uses the variables of Thiele, then $\lambda_x{}^0(t)$ and $\lambda_y{}^0(t)$ are also not identically zero.

APPENDIX.

On a Theorem in the Pfaffian Dynamics of Birkhoff.

The conservative Hamiltonian principle

$$\delta \int \left\{ \sum_{i=1}^{n} p_i \dot{q}_i - H(p_1, \cdots, q_n) \right\} dt = 0 \text{ or } \dot{q}_i = \partial H/\partial p_i, \ -\dot{p}_i = \partial H/\partial q_i, \quad (i = 1, \cdots, n)$$

which is rather unsymmetrical with respect to q_i and p_i has been generalized by Birkhoff * to the "Pfaffian" principle

$$(1) \qquad \delta \int \left\{ \sum_{i=1}^{2n} R_i dx_i - H dt \right\} = 0, \quad \text{i. e.} \quad \delta \int \left\{ \sum_{i=1}^{2n} R_i \dot{x}_i - H \right\} dt = 0$$

or

$$(2) \qquad \sum_{j=1}^{2n} (\partial R_i/\partial x_j - \partial R_j/\partial x_i) \dot{x}_j = \partial H/\partial x_i; \qquad (i = 1, 2, \cdots, 2n-1, 2n)$$

where

$$(3) \qquad H = H(x_1, \cdots, x_{2n}); \qquad R_i = R_i(x_1, \cdots, x_{2n})$$

are $2n + 1$ given functions of the $2n$ coördinates of the phase space and

$$(4) \qquad \det (\partial R_i/\partial x_j - \partial R_j/\partial x_i) \neq 0.$$

The equations (2) which possess the integral $H = \text{const.}$ are, according to (1), invariant for *any* transformation of the coördinates. The conservative Hamiltonian systems are special cases of the symmetrical Pfaffian problem (2), the condition (4) being for

$$(5) \qquad x_i = q_i, \quad x_{i+n} = p_i, \quad R_i = x_{i+n}, \quad R_{i+n} = 0; \qquad (i = 1, 2, \cdots, n)$$

obviously fulfilled. Birkhoff has shown that the majority of the essential properties of the Hamiltonian systems hold also for the Pfaffian case. For the purposes of Birkhoff a theorem concerning the characteristic exponents of the variational equations belonging to a given periodic solution

$$(6) \qquad x_i = x_i(t); \qquad (i = 1, 2, \cdots, 2n-1, 2n)$$

of (2) is of particular importance. This theorem has been demonstrated

---

* G. D. Birkhoff, "Dynamical Systems," *American Mathematical Society Colloquium Publications*, Vol. 9 (1927), p. 55, p. 89 etc. Cf. also L. Féraud, "On Birkhoff's Pfaffian Mechanics," *Transactions of the American Mathematical Society*, Vol. 32 (1930), p. 817 etc.

by Birkhoff * only under certain restrictions. In the present note a simple proof is given which holds without any particular assumptions, showing that the theorem of Birkhoff is a general property of all conservative Pfaffian systems.

The variational equations belonging to the given periodic solution (6) of (2), namely the $2n$ linear differential equations

$$(7)\qquad \sum_{j=1}^{2n}(\partial R_i/\partial x_j - \partial R_j/\partial x_i)\dot{\xi}_j + \sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)\xi_l\dot{x}_j = \sum_{j=1}^{2n}\partial^2 H/\partial x_j\partial x_i\xi_j,$$
$$(i = 1, 2, \cdots, 2n-1, 2n),$$

can be written in the form

$$(8)\qquad \dot{\xi}_i = \sum_{j=1}^{2n} a_{ij}(t)\xi_j; \qquad (j = 1, 2, \cdots, 2n-1, 2n)$$

inasmuch as the "velocities" $\dot{x}_j$ and the partial derivatives occuring in (7) are, by virtue of (6) and (3), given functions of t (the determinant of the coefficients

$$(9)\qquad \partial R_i/\partial x_j - \partial R_j/\partial x_i = s_{ij}(t)$$

is, according to (4), different from zero). If one uses instead of ξ the letter η and interchanges i and j equations (7) can be written in the form

$$(10)\qquad \sum_{i=1}^{2n}(\partial R_i/\partial x_j - \partial R_j/\partial x_i)\dot{\eta}_i + \sum_{i=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)\eta_l\dot{x}_i = -\sum_{i=1}\partial^2 H/\partial x_i\partial x_j\eta_i,$$
$$(j = 1, 2, \cdots, 2n-1, 2n).$$

Furthermore, the expression

$$(11)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{i=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)(\xi_j\eta_i\dot{x}_l - \xi_l\eta_i\dot{x}_j - \xi_j\eta_l\dot{x}_i)$$

is the sum six of six trilinear forms three of which are the negatives of the three others. Multiplying (7) by η_i and (10) by ξ_j one obtains $4n$ relations and on adding these it follows that the expression

$$(12)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}(\partial R_i/\partial x_j - \partial R_j/\partial x_i)(\dot{\xi}_j\eta_i + \dot{\eta}_i\xi_j) + \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)(\xi_l\eta_i\dot{x}_j + \xi_j\eta_l\dot{x}_i)$$

* *Loc. cit.*, p. 90-91.

also vanishes. The sum

$$(13)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n}(\partial R_i/\partial x_j - \partial R_j/\partial x_i)(\dot{\xi}_j\eta_i + \dot{\eta}_i\xi_j) + \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(\partial^2 R_i/\partial x_l\partial x_j - \partial^2 R_j/\partial x_l\partial x_i)\xi_j\eta_i\dot{x}_l$$

of the two vanishing expressions (11), (12) is, according to (9), simply the time derivative of the bilinear covariant

$$(14)\qquad \sum_{i=1}^{2n}\sum_{j=1}^{2n} s_{ij}(t)\xi_j(t)\eta_i(t),$$

associated with the Pfaffian

$$(1')\qquad \sum_{i=1}^{2n} R_i dx_i - H dt.$$

Since the expression (13) is zero the value of (14) is, for two arbitrary solutions ξ, η of the equations of variations belonging to (6), independent of t.

It obviously follows, in the same manner as above, that the characteristic equation associated with (7) or (8) is always a reciprocal one. This is the theorem of Birkhoff in its generalized form. The skew symmetrical, non-singular bilinear form (14) is, according to (5) and (9), the Pfaffian generalization of the bilinear differential invariant of Poincaré used above which corresponds to the integral invariant

$$\int\int \sum_{i=1}^{n} \delta p_i \delta q_i$$

of the Hamiltonian systems. For the integral invariants of the Pfaffian systems cf., for instance, the paper of Féraud, *loc. cit.*

THE JOHNS HOPKINS UNIVERSITY.

THE EQUATION OF STABILITY OF PERIODIC ORBITS OF THE RESTRICTED PROBLEM OF THREE BODIES IN THIELE'S REGULARISING COÖRDINATES.

By Jenny E. Rosenthal.*

The harmonic analysis of periodic orbits calculated by mechanical quadrature at the Copenhagen Observatory is available in print in some of its main features; † the rest has not yet been published. Since the harmonic analysis has been done mostly for ejection orbits the Fourier series also have been calculated in Thiele's variables. A determination of the stability character of the orbits in the Copenhagen material would make it necessary, therefore, to write the Jacobi second order differential equation explicitly in terms of Thiele's variables. This equation then determines the characteristic exponents. The preceding article by A. Wintner ‡ includes a study of the Jacobi differential equation for any mechanical problem with two degrees of freedom. This article has also a discussion of the connection between this second order equation and the original fourth order variational equations. On the basis of formulas given there I have expressed the Jacobi second order differential equation of the restricted problem of three bodies for an arbitrary value of the mass ratio in terms of Thiele's regularising variable. The regularising transformation in the general case was first given by Burrau § (Thiele has published his regularising transformation only in the case of two equal masses.)

The regularised differential equations are as follows:

$$(1) \qquad E'' - \lambda F' = \partial\Omega/\partial E; \quad F'' + \lambda E' = \partial\Omega/\partial F; \quad (E')^2 + (F')^2 - 2\Omega = 0,$$

* National Research Fellow.

† E. Strömgren, *Tre Aartier Celest Mekanik paa Københavns Observatorium*, pp. 46-86, Copenhagen 1923.

‡ A Wintner, "Three Notes on Characteristic Exponents and Equations of Variation in Celestial Mechanics," *American Journal of Mathematics*, Vol. 53 (1931), p. 605.

§ C. Burrau, "Über einige in Aussicht genommene Berechnungen betreffend einen Spezialfall des Dreikörper-Problems," *Vierteljahresschrift der Astronomischen Gesellschaft*, Vol. 41 (1906), pp. 261 ff. Cf. also J. Fischer-Petersen, "Über unendlich kleine periodische Bahnen um die Massenpunkte im problème restreint," *Astronomische Nachrichten*, Vol. 200 (1915), pp. 387-388. [Publikationer fra Köbenhavns Observatorium, No. 22].

where

(2) $$2\lambda = \cos 2iF - \cos 2E,$$

(3) $$\Omega = (1/16)(\cos 4iF - \cos 4E) - (k/4)(\cos 2iF - \cos 2E) + 8\cos iF + (1/4)(1-2\mu)(\cos E \cos 3iF - \cos 3E \cos iF - 32 \cos E).$$

Here k is the energy constant, μ the mass ratio, E and F the cartesian coördinates in the plane of Thiele's variables and the prime represents differentiation with respect to Thiele's time variable ψ (and not with respect to the ordinary time t).

Let

(4) $$E = E(\psi),\ F = F(\psi)$$

be a given solution of (1); and let δE and δF be such functions of ψ that the equations:

(5) $$E = E(\psi) + \delta E;\ F = F(\psi) + \delta F$$

represent a virtual motion infinitesimally close to the given orbit (4) in such a manner that this variation, i. e., the transition from (4) to (5), be isoenergetic. This assumption, namely $\delta k = 0$, is necessitated by the third equation (1). In Wintner's article it is generally shown that the non trivial (i. e., the non identically vanishing) characteristic exponent (in other words, the stability character) is determined in any case by such an isoenergetic transformation ($\delta k = 0$). The direction cosines of the normal of orbit (4) in the E, F plane are:

(6) $$-F'[(E')^2 + (F')^2]^{-1/2},\quad +E'[(E')^2 + (F')^2]^{-1/2}.$$

The normal displacement is therefore

(7) $$-F'[(E')^2 + (F')^2]^{-1/2}\delta E + E'[(E')^2 + (F')^2]^{-1/2}\delta F,$$

or

(8) $$\theta[(E')^2 + (F')^2]^{-1/2} \text{ where } \theta = E'\delta F - F'\delta E.$$

satisfies the following differential equation

(9) $$\alpha(\psi)\theta'' + \beta(\psi)\theta' + \gamma(\psi)\theta = 0.$$

Here

$$\alpha(\psi) = [(E')^2 + (F')^2],$$
$$\beta(\psi) = b_1E' + b_2F',$$
$$\gamma(\psi) = c_0 + c_1E' + c_2F' + [(E')^2 + (F')^2]\,[c_3 + c_4E' + c_5F'],$$

where

$$b_1 = b_{10} + b_{11}(1 - 2\mu), \qquad b_2 = b_{20} + b_{21}(1 - 2\mu),$$
$$c_0 = c_{00} + c_{01}(1 - 2\mu) + c_{02}(1 - 2\mu)^2,$$
$$c_1 = c_{10} + c_{11}(1 - 2\mu), \qquad c_2 = c_{20} + c_{21}(1 - 2\mu),$$
$$c_3 = c_{30} + c_{31}(1 - 2\mu), \qquad c_4 = c_{40}, \qquad c_5 = c_{50},$$

and

$$b_{10} = (1/2)(-\sin 4E + 2k \sin 2E),$$
$$b_{11} = (1/2)(\sin E \cos 3iF - 3 \sin 3E \cos iF - 32 \cos E),$$
$$b_{20} = (i/2)(\sin 4iF - 2k \sin 2iF + 32 \sin iF),$$
$$b_{21} = (i/2)(3 \cos E \sin 3iF - \cos 3E \sin iF),$$

$$\begin{aligned} c_{00} = (1/16)\,[&(\cos 8iF - \cos 8E) - 4k\,(\cos 6iF - \cos 6E) + 64 \cos 5iF \\ &+ 4k^2\,(\cos 4iF - \cos 4E) - 64(1 + 2k) \cos 3iF \\ &+ (1024 + 4k) \cos 2iF - 4k \cos 2E + 128k \cos iF - 1024], \end{aligned}$$

$$\begin{aligned} c_{01} = (i/8)\,[&-32 \cos 5E + 64(1 + k) \cos 3E - 64k \cos E \\ &- 3\,(\cos 7E \cos iF - \cos E \cos 7iF) + 6k\,(\cos 5E \cos iF \\ &- \cos E \cos 5iF) + (\cos 5E \cos 3iF - \cos 3E \cos 5iF) \\ &- 2k\,(\cos 3E \cos iF - \cos E \cos 3\,iF) - 32 \cos 3E \cos 2iF \\ &+ 96 \cos E \cos 4iF - 96 \cos E \cos 2iF], \end{aligned}$$

$$\begin{aligned} c_{02} = (1/16)\,[&5\,(\cos 6iF - \cos 6E) - 64 \cos 3iF + 5 \cos 2iF \\ &- 1029 \cos 2E + 4(\cos 2E \cos 6iF - \cos 6E \cos 2iF) \\ &- 6(\cos 2E \cos 4iF - \cos 4E \cos 2iF) + 192 \cos 2E \cos iF \\ &+ 64 \cos 2E \cos 3iF - 192 \cos 4E \cos iF], \end{aligned}$$

$$\begin{aligned} c_{10} = (3i/4)\,[&\sin 6iF - 2k \sin 4iF + 32 \sin 3iF + \sin 2iF - 32 \sin iF \\ &- 2 \cos 2E \sin 4iF + 4k \cos 2E \sin 2iF - 64 \cos 2E \sin iF], \end{aligned}$$

$$\begin{aligned} c_{11} = (3i/4)\,[&\cos 5E \sin iF - 4 \cos 3E \sin 3iF + \cos 3E \sin iF \\ &+ 3 \cos E \sin 5iF - 3 \cos E \sin 3iF + 4 \cos E \sin iF], \end{aligned}$$

$$\begin{aligned} c_{20} = (3/4)\,[&-\sin 6E + 2k \sin 4E - \sin 2E + 2 \sin 4E \cos 2iF \\ &- 4k \sin 2E \cos 2iF], \end{aligned}$$

$$\begin{aligned} c_{21} = (3/4)\,[&-32 \sin 3E + 32 \sin E - 3 \sin 5E \cos iF + 4 \sin 3E \cos 3iF \\ &+ 3 \sin 3E \cos iF - \sin E \cos 5iF - \sin E \cos 3iF \\ &+ 64 \sin E \cos 2iF - 4 \sin E \cos iF], \end{aligned}$$

$$\begin{aligned} c_{30} = (1/2)\,[&10 + 3\,(\cos 4iF + \cos 4E) + 2k\,(\cos 2iF + \cos 2E) \\ &- 16 \cos iF - 20 \cos 2iF \cos 2E], \end{aligned}$$

$$c_{31} = -2\cos E \cos 3iF - 2\cos 3E \cos iF - 8\cos E,$$
$$c_{40} = -2i \sin 2iF,$$
$$c_{50} = -2\sin 2E,$$

In these expressions for the three coefficients, E, F, E' and F' are to be considered as given functions of ψ on account of (4). The function (4) [and its derivatives] are given numerically by Strömgren (*loc. cit.*) in the form of Fourier series. Substitution of Strömgren's expressions for E and F in the above given expressions for α, β and γ determines the differential equation (9) which corresponds to the given orbit. The further treatment of (9) is done by known methods, for example by Hill's method.

In researches at the Copenhagen Observatory Burrau * has treated such virtual displacements. A similar arrangement of the calculations would probably make the seemingly complicated formulas given above applicable for numerical purposes. Numerical calculations are made more practicable by the fact that in a given stage of a given group it is often possible to predict what terms will make a contribution which is numerically negligible. It may be stressed that the above expressions for α, β, γ are valid without the neglect of any terms; the length of the expressions seems to be unavoidable. If the Fourier series of solution (4) are known analytically so that the Fourier coefficients and the period are given power series of Jacobi's constant k, it is simple to determine in advance the preponderant terms. Such convergent analytical expressions are known, for example,† in the neighborhood of the masses for groups f and g and at a large distance from the masses for groups l and m; and it is easy to obtain corresponding developments for the libration groups, a, b, c, d, e in the neighborhood of the corresponding libration point. If we are concerned with periodic solutions (of these groups) whose range is neither quite in the neighborhood of the masses or the libration points nor in the neighborhood of the infinitely distant point, the analytical developments are not applicable, so that we have to use the numerical Fourier series given by E. Strömgren (*loc. cit.*). This is also the case for groups k, n, etc., for which there is no analytical theory available at present. For the rôle of differential equation (9) of the characteristic exponent in the theory of the Strömgren groups see Wintner (*loc. cit.*). From the expressions

* C. Burrau, "Recherches numériques concernant les solutions périodiques d'un cas spécial du problème des trois corps (Deuxième mémoire)," *Astronomische Nachrichten*, Vol. 136 (1894), p. 136.

† For reference cf. E. Strömgren, *loc. cit.*, pp. 70-71. Cf. also E. Strömgren, "Forms of Periodic Motion in the Restricted Problem etc.," *Publikationer of mindre Meddeleser fra Köbenhavns Observatorium*, No. 39.

above it is clear that the formulas are considerably simplified in the symmetrical case ($\mu = 1/2$) since then all terms with $(1-2\mu)$ and $(1-2\mu)^2$ as factors drop out. It may be mentioned here that, of course, the characteristic exponents can be determined by the same method in the system of variables x, y, t (instead of E, F, ψ), and in that case the expressions for α, β, γ are not so lengthy.* But in the system x, y, t one cannot treat for instance even the essential stages of the group, such as the passing of a group through an ejection orbit. Moreover, Strömgren's harmonic analysis (*loc. cit.*) has always been done in the system E, F, ψ and possible sources of error would arise in reducing Strömgren's Fourier series (as long as they do not belong to the ejection orbit) to the x, y, t system. (It would also be necessary then to determine each time the period *referred to* t which would mean considerable additional numerical work for single orbits and might give rise to new sources of error since the connection between t and ψ is given explicitly only by a differential equation.†) Hence there is apparently no way of avoiding the above expressions for α, β, γ in problems connected with characteristic exponents of the Copenhagen groups [compare also Wintner (*loc. cit.*)].

In conclusion I wish to thank Dr. A. Wintner for his suggestions and advice.

THE JOHNS HOPKINS UNIVERSITY.

* A. Wintner, "Über die Jacobische Differentialgleichung des restringierten Dreikörperproblems," *Sitzungsberichte der mathematisch-physikalischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig*, Vol. 82 (1930), pp. 345-354.

† Cf. A. Wintner, "Über eine Revision der Sortentheorie des restringierten Dreikörperproblems," *ibid.*, p. 46.

ALGEBRAS OF CERTAIN DOUBLY TRANSITIVE GROUPS.

By R. D. Carmichael.

A class of finite algebras $A[p^n]$ is defined directly (§ 1) by means of doubly transitive groups of prime-power degree p^n and order $p^n(p^n-1)$ and is shown (§ 1) to be equivalent to a class of finite algebras defined by Dickson in 1905 (Göttingen Nachrichten, 1905). The set of all linear transformations on the marks of an $A[p^n]$ induces on those marks a group which is conjugate to that by which the algebra is defined (§ 2). Three forms are given (§ 3) to the (only partially solved) problem of determining all algebras $A[p^n]$, one of them being of fundamental importance in the investigation of the group of isomorphisms of an Abelian group of order p^n and type $(1, 1, \cdots, 1)$. This problem deserves further attention. Two algebras $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic (§ 4) when and only when their multiplicative groups are simply isomorphic. The integral elements of an $A[p^n]$ form (§ 5) a Galois field $GF[p]$. The algebras $A[p^n]$ are capable (§ 6) of various analytical representations, including as a special case that employed by Dickson. A large class of doubly transitive groups of degree p^n and order $p^n(p^n-1)$ is exhibited (§ 7) and these groups are employed (§ 8) in the rapid construction of a large class of algebras $A[p^n]$, closely related to those determined by Dickson by other methods.

1. *Construction of Algebras* $A[s]$. Let G be a doubly transitive group of degree ρ and order $\rho(\rho-1)$. Then it is well known that ρ is a prime-power p^n $(n \geqq 1)$, that such a doubly transitive group G contains a single subgroup H of order p^n, that this Sylow subgroup H is Abelian and of type $(1, 1, \cdots, 1)$, that H contains all the elements of G (p^n-1 in number) each of which displaces all the symbols permuted by G, that H is self-conjugate in G, and that every element in G and not in H is a regular permutation on just p^n-1 symbols.

Let $a_0, a_1, a_2, \cdots, a_{s-1}$, where $s = p^n$, be the p^n symbols permuted by G. Then H permutes these symbols among themselves according to a regular group, as is well known and may be readily shown from the fact that H consists of the identity and p^n-1 elements each of which permutes all the symbols. Then there is one and just one element h_i of H which replaces a_0 by a_i.

Let us denote by M the subgroup of order p^n-1 in G each element

of which leaves a_0 fixed. It is a regular group on $a_1, a_2, \cdots, a_{s-1}$. Hence there is one and just one element m_i of M which replaces a_1 by a_i. It is evident that $m_i^{-1}h_1m_i = h_i$.

By means of these properties of G we shall define an algebra $A[p^n]$. Let the p^n symbols or marks of this algebra be denoted by $u_0, u_1, u_2, \cdots, u_{s-1}$. We introduce a law of addition for the marks u_i of this algebra in the following manner: The sum $u_i + u_j$ is the mark u_k $(u_i + u_j = u_k)$ where k is such that $h_ih_j = h_k$ in the group G. Then, in particular, $u_i + u_0 = u_i$ for every mark u_i.

For the purpose of defining a law of multiplication for the marks u_i, exclusive of the zero-mark u_0, we employ the elements of the subgroup M of G. We write $u_iu_j = u_l$ $(i > 0,\ j > 0)$ where l is defined by the relation $m_l^{-1} = m_i^{-1}m_j^{-1}$ or $m_l = m_jm_i$. [At this point it would seem more natural to take $u_iu_j = u_\lambda$ where $m_\lambda = m_im_j$; but this would give $(b + c)a = ba + ca$ instead of the relation $a(b + c) = ab + ac$, presently to be established; and the latter relation is slightly more natural from the point of view of the algebras.] We define the products u_iu_0 and u_0u_i by the requirement that each of them shall have the value u_0.

With the named laws of addition and multiplication the marks $u_0, u_1, \cdots, u_{s-1}$ constitute an algebra of the type defined by Dickson * in 1905, as we shall now show. Dickson subjects his algebras to nine postulates. The first four of these postulates require merely that all the marks of the algebra shall form a group under addition. The next four postulates require merely that all the marks, exclusive of the zero-mark, shall form a group under multiplication. The remaining postulate asserts that if a, b, c are elements of the algebra then $a(b + c) = ab + ac$. With these postulates in hand it is easy to show (cf. Dickson, l. c.) that the additive group and the multiplicative group have the properties already employed. Therefore, in order to show that the algebras here defined are identical with those of Dickson, it is sufficient to prove that his last postulate is verified.

As expressed in terms of the u's we have then to establish the following relation:

$$u_i(u_\rho + u_\sigma) = u_iu_\rho + u_iu_\sigma. \tag{1.1}$$

This is immediately verified if any one of the subscripts i, ρ, σ is zero. Then for the further argument suppose that each of them is greater than zero. Since (1.1) involves two operations it is convenient to reduce the relation to be proved to a corresponding relation among the elements of H and M

---

* Dickson, *Göttinger Nachrichten*, 1905.

since they are all subject to the single rule of combination in G. From the definitions of addition and multiplication we have the following propositions:

$$u_\rho + u_\sigma = u_\tau \quad \text{if} \quad h_\rho h_\sigma = h_\tau, \qquad u_i u_\rho = u_\lambda \quad \text{if} \quad m_\rho m_i = m_\lambda,$$
$$u_i u_\sigma = u_\nu \quad \text{if} \quad m_\sigma m_i = m_\nu, \qquad u_i u_\tau = u_\alpha \quad \text{if} \quad m_\tau m_i = m_\alpha,$$
$$u_\lambda + u_\nu = u_\beta \quad \text{if} \quad h_\lambda h_\nu = h_\beta.$$

In order to establish the required relation (1.1) it is necessary and sufficient to show that $\alpha = \beta$. Now we have

$$\begin{aligned} h_\beta &= h_\lambda h_\nu = m_\lambda^{-1} h_1 m_\lambda \cdot m_\nu^{-1} h_1 m_\nu \\ &= (m_\rho m_i)^{-1} h_1 (m_\rho m_i) \cdot (m_\sigma m_i)^{-1} h_1 (m_\sigma m_i) \\ &= m_i^{-1} h_\rho h_\sigma m_i = m_i^{-1} h_\tau m_i \\ &= m_i^{-1} m_\tau^{-1} h_1 m_\tau m_i = (m_\tau m_i)^{-1} h_1 (m_\tau m_i) \\ &= m_\alpha^{-1} h_1 m_\alpha = h_\alpha. \end{aligned}$$

Since $h_\beta = h_\alpha$ it follows that $\alpha = \beta$ and hence that (1.1) is established.

From the foregoing analysis it follows that every doubly transitive group of prime-power degree p^n $(n \geqq 1)$ and order $p^n(p^n - 1)$ may be employed for the definition of an algebra $A[p^n]$. In the next section we consider the converse problem.

2. *Linear Transformations in an* $A[s]$. If β is any given one of the marks $u_0, u_1, \cdots, u_{s-1}$ of an $A[s]$, $s = p^n$, and if x is a variable running over the marks of the algebra then $x + \beta$ is a new variable x' running over the marks of the algebra. Thus we have the transformation $x' = x + \beta$ corresponding to the addition of β on the right to all the marks of the algebra. By varying β we obtain the p^n transformations corresponding to the additive group of the algebra.

More generally the set of all transformations

$$(2.1) \qquad x' = \alpha x + \beta,$$

where α and β run independently over all the marks of the algebra except that α remains different from the zero mark u_0, constitutes a group K, as one may readily verify by means of the stated properties of $A[s]$. Its order is $s(s-1)$. Each element of K permutes the marks of $A[s]$ among themselves; thus K gives rise to a permutation group K_1 of degree s on the marks $u_0, u_1, \cdots, u_{s-1}$. If a and b are any two distinct marks of $A[s]$ then the marks u_0, u_1, as values of x, are replaced by the marks a, b respectively, as values of x', by the transformation

$$x' = (b - a)x + a.$$

Therefore K_1 is a doubly transitive group of degree s and order $s(s-1)$.

From this it follows that *the number s of marks in an algebra $A[s]$* (satisfying Dickson's postulates) *is necessarily of the form $s = p^n$*, a result proved by Dickson (*l. c.*) directly from his postulates.

Those transformations (2.1) in which $\alpha = u_1$ correspond to the additive group of the algebra, since they may be written in the form $x' = x + \beta$. [Thence follow readily the known essential properties of the additive group of the algebras $A[s]$.] Those transformations (2.1) in which $\beta = u_0$ may be written in the form $x' = \alpha x$. They correspond to the multiplicative group of the algebra, the element $x' = \alpha x$ corresponding to multiplication on the right by α. This multiplicative group induces on the non-zero marks of the algebra a regular permutation group of order $p^n - 1$.

Let M denote the permutation group on the marks $u_1, u_2, \cdots, u_{s-1}$ induced by the multiplicative group of the algebra and denote by m_i the element of M induced by the transformation $x' = u_i x$, $u_i \neq u_0$. Let H denote the permutation group on the marks $u_0, u_1, \cdots, u_{s-1}$ induced by the additive group of the algebra and denote by h_i the element of H induced by the transformation $x' = x + u_i$. Let G be the group generated by H and M. Then h_i replaces $x = u_0$ by $x' = u_i$ and m_i replaces $x = u_1$ by $x' = u_i$; whence it follows that $h_i = m_i^{-1} h_1 m_i$. Then $u_i + u_j = u_k$ where k is such that $h_k = h_i h_j$, while $u_i u_j = u_l$ where l is such that $m_l^{-1} = m_i^{-1} m_j^{-1}$.

From these results it follows that the group G to which the algebra leads by use of (2.1) may in turn be employed as in § 1 to recover the algebra itself. Therefore, *every possible algebra $A[s]$, satisfying Dickson's postulates, is an algebra $A[p^n]$ defined as in § 1 by means of a doubly transitive group of prime-power degree p^n ($n \geqq 1$) and order $p^n(p^n - 1)$ while conversely such a doubly transitive group is induced by the totality of transformations of the form* (2.1) *on the marks of such an algebra.*

3. *Three Equivalent Forms of an Unsolved Problem.* From the theorem just stated it follows that the problem of constructing all algebras $A[p^n]$ is equivalent to the problem of constructing all doubly transitive groups of degree p^n and order $p^n(p^n - 1)$. With respect to these latter groups it is not difficult to establish the following theorem:

Every doubly transitive group G of degree p^n and order $p^n(p^n - 1)$ is contained in the holomorph of the Abelian group H of order p^n and type $(1, 1, \cdots, 1)$ when that holomorph is written in the usual way as a permutation group. Moreover, the regular subgroup M of G, consisting of those elements which leave one symbol fixed, is contained in the group I of isomorphisms of H.

It is easy also to establish the following theorem:

For every regular group I_1 of degree and order $p^n - 1$ contained in the group I of isomorphisms of an Abelian group of order p^n and type $(1, 1, \cdots, 1)$ there exists one and just one doubly transitive group G of degree p^n and order $p^n(p^n - 1)$ containing I_1 as a subgroup.

From these results follows readily the theorem (proved otherwise by Dickson, *l. c.*) that *the multiplicative group of an algebra $A[p^n]$ is simply isomorphic with a regular subgroup I_1 of degree $p^n - 1$ contained in the group I of isomorphisms (with itself) of an Abelian group H of order p^n and type $(1, 1, \cdots, 1)$ when I is represented in the usual way as a permutation group on the elements of H exclusive of the identity.*

These considerations lead to the formulation of the following three problems:

1. To construct all the regular subgroups I_1 of degree $p^n - 1$ contained in the group I of isomorphisms (with itself) of an Abelian group H of order p^n and type $(1, 1, \cdots, 1)$ when I is represented in the usual way as a permutation group on the elements of H exclusive of the identity.

2. To construct all doubly transitive groups of degree p^n and order $p^n(p^n - 1)$.

3. To construct all algebras $A[p^n]$ subject to the postulates of Dickson (*l. c.*).

From foregoing results it follows that the solution of any one of these problems carries with it the solution of the other two. It appears that, up to the present, no one of these problems has been completely solved. The work of Dickson (*l. c.*) is the most important which has yet been done in this direction. His most comprehensive results, however, are based on an empirical (unproved) proposition; though this proposition is a remarkable one if true (and he has verified it in a wide range of cases), nevertheless no one else (so far as I know) has given it further consideration. It appears that the problem here formulated is a difficult one; its importance is indicated by the three-fold formulation and the variety of connections which it is thus shown to have. It deserves further attention.

For every value of p^n, as is well known, there exists a cyclic I_1; every Abelian I_1 is cyclic. When this I_1 is employed, the resulting algebra $A[p^n]$ is the Galois field $GF[p^n]$. But, so far as I am aware, no direct proof has been given of the existence of this cyclic I_1; that is to say, the known proofs appear all to be based on the (previously proved) existence of the $GF[p^n]$

or on what is essentially equivalent to that, such existence being established by methods which are not directly group-theoretic in character. It is not satisfactory thus to be driven outside the domain of direct group-theoretic considerations to establish the existence of this cyclic I_1. Emphasis is here put upon this particular problem in the hope of directing to it the attention of other investigators.

4. *Simple Isomorphism of Algebras* $A[p^n]$. Two algebras $A_1[p^n]$ and $A_2[p^n]$ will be called simply isomorphic if each element of A_1 may be made to correspond uniquely to an element of A_2 in such a way that each element of A_2 is the correspondent of a single element of A_1 while moreover the sum [product] of any two elements in A_1 corresponds to the sum [product] of the corresponding two elements of A_2. It will be said that two simply isomorphic algebras are identical. Any two algebras $A[p^n]$ are evidently such that their additive groups are simply isomorphic. An obvious necessary condition for the algebras to be simply isomorphic is that their multiplicative groups of order $p^n - 1$ shall be simply isomorphic. We shall show that this condition is also sufficient.

If the multiplicative groups of $A_1[p^n]$ and $A_2[p^n]$ are simply isomorphic then the doubly transitive groups of degree p^n and order $p^n(p^n - 1)$, to which they lead by the method of § 2, have simply isomorphic regular sub-groups of degree $p^n - 1$, as is seen from the named isomorphism of the multiplicative groups of the algebras. Hence these two doubly transitive groups are conjugate. Now, on recovering the algebras from these conjugate groups, by the method of § 1, we exhibit the algebras themselves as simply isomorphic.

Thus we have the following theorem:

Two algebras $A_1[p^n]$ *and* $A_2[p^n]$ *are simply isomorphic when and only when their multiplicative groups are simply isomorphic.*

5. *Integral Elements of an Algebra* $A[p^n]$. Denote the elements of an $A[p^n]$, as before, by the symbols $u_0, u_1, \cdots, u_{s-1}$. An element of the form $u_1 + u_1 + \cdots + u_1$ will be called an integral element; the other elements are said to be non-integral. From the properties of the additive group H of the algebra it follows that there are just p integral elements of the algebra. When there is no danger of confusion these may be denoted by $0, 1, \cdots, p - 1$, where 0 and 1 denote the elements u_0 and u_1 respectively. Addition and multiplication of the integral elements are equivalent to ordinary addition and multiplication followed by a reduction modulo p. Hence the integral elements of $A[p^n]$ form a sub-algebra which is simply isomorphic with $GF[p]$.

In particualr, an algebra $A[p]$ consists entirely of integral elements and is the $GF[p]$.

It is well known that $GF[p^n]$ contains a subfield $GF[p^k]$ when and only when k is a factor of n. If an $A[p^n]$ contains a sub-algebra $A[p^k]$, then the multiplicative group of order $p^k - 1$ of the latter must be a subgroup of the multiplicative group of order $p^n - 1$ of the former: hence $p^k - 1$ must be a factor of $p^n - 1$, whence it follows that k is a factor of n.

6. *Analytical Representation of Algebras* $A[p^n]$. Let us write $n = k\nu$ where k and ν are positive integers (either or both of which may be unity). We now denote the p^n elements of an algebra $A[p^n]$ by $(a_1, a_2, \cdots, a_k)$ where the a's run independently over the marks of the $GF[p^\nu]$. In view of the properties of the additive group H of the algebra it is evident that we may take for the rule of addition in the algebra that expressed by the formula

$$(6.1) \qquad (a_1, a_2, \cdots, a_k) + (b_1, b_2, \cdots, b_k) = (a_1 + b_1, \cdots, a_k + b_k).$$

Then $(0, 0, \cdots, 0)$ is the zero element of the algebra. The product of the zero element by any other element (in either order) is the zero element. It remains to define a suitable rule of multiplication for the non-zero elements of the algebra.

The multiplication of the non-zero elements is according to a group M which permutes these non-zero elements according to a regular group contained in the group I of isomorphisms of H with itself. Moreover (§ 2) the group of linear transformations in the algebra permutes the marks of the algebra according to a doubly transitive group of degree p^n and order $p^n(p^n - 1)$. From these facts and from the analytical representation T of the group I given in an earlier memoir * it follows that if

$$(6.2) \qquad (a_1, a_2, \cdots, a_k) \cdot (x_1, x_2, \cdots, x_k) = (x_1', x_2', \cdots, x_k')$$

then we have

$$(6.3) \qquad x_i' = \sum_{t=1}^{\nu} \sum_{j=1}^{k} a_{ijt} x_j^{p^{n-t}}, \qquad (i = 1, 2, \cdots, k),$$

where the coefficients a_{ijt} are marks of $GF[p^\nu]$ which depend on $(a_1, a_2, \cdots, a_k)$ but are independent of $(x_1, x_2, \cdots, x_k)$. Consequently the multiplicative group of the algebra may be defined by means of a transformation group whose elements have the foregoing form. A necessary and sufficient condition on these transformations is that they shall permute the non-zero marks of the algebra according to a regular group.

---

* Carmichael, *American Journal of Mathematics*, Vol. 52 (1930), pp. 745-788 (see § 8).

When $\nu = 1$ the transformation group is linear and we have the form of analytical representation employed by Dickson (*l. c.*). When $k = 1$ we have the other extreme case of the foregoing transformations. In this case we have $\nu = n$ and the marks of the algebra are the symbols (a) where a runs over the marks of $GF[p^n]$. The rule of addition in the algebra, namely, $(a) + (b) = (a + b)$, coincides with the rule of addition in $GF[p^n]$. For the product $(a)(x)$ we have $(f(a, x))$ where $f(a, x)$ has the form

$$(6.4) \qquad f(a, x) = \sum_{t=1}^{n} a_t x^{p^{n-t}}.$$

Therefore we may write

$$(6.5) \quad (\alpha_i)(x) = (f(\alpha_i, x)) = (\sum_{t=1}^{n} a_t^{(i)} x^{p^{n-t}}), \qquad (i = 0, 1, \cdots, p^n - 2),$$

where the α's are the non-zero marks of $GF[p^n]$ and the $a_t^{(i)}$ are marks of $GF[p^n]$ to be suitably determined. We take (1) to be the unit element in the algebra. Then we have

$$a_1^{(i)} + a_2^{(i)} + \cdots + a_n^{(i)} = \alpha_i.$$

We have also $(0)(x) = (f(0, x)) = (0)$.

It thus appears that every algebra $A[p^n]$ may be represented analytically by means of $GF[p^n]$. As already indicated, the problem of determining all such algebras has not yet been completely solved. In § 8 we shall employ the method just indicated to set forth the analytical representations of each of a large class of algebras $A[p^n]$.

It is convenient to close this section with the statement of three propositions whose proofs will be omitted. If p is an odd prime the multiplicative group M of an algebra $A[p^n]$ contains just one element of order 2 (perhaps most readily proved by aid of (6.3) with $\nu = 1$). In the multiplicative group M of an algebra $A[p^n]$ the Sylow subgroups of odd order are cyclic and those of even order are either cyclic or of the sole non-cyclic type containing a single element of order 2. If M contains a non-cyclic Sylow subgroup of order 2^a then this Sylow subgroup contains at least three subgroups of each of the orders $2^2, 2^3, \cdots, 2^{a-1}$.

7. *Certain Doubly Transitive Groups of Degree p^n.* In proceeding to construct algebras $A[p^n]$ it is convenient first to consider certain doubly transitive groups which are representable by means of transformations of the form

$$(7.1) \qquad x' = ax^{p^t} + b, \qquad a \neq 0,$$

where a and b belong to $GF[p^n]$, t belongs to the set $0, 1, \cdots, n-1$ and

x and x' are variables running over the marks of $GF[p^n]$. The permutation groups involved are those according to which the marks of $GF[p^n]$ are permuted by the named transformation groups. When a, b, t range respectively over all the elements on which they may range we have a doubly transitive group of degree p^n and order $p^n(p^n-1)n$. The transformations

$$(7.2) \qquad x' = x + 1, \qquad x' = \omega x, \qquad x' = x^{p^\alpha},$$

where ω is a primitive mark of the field and α is a factor n, generate a group of order $p^n(p^n-1)n/\alpha$ whose elements are all the elements of the form (7.1) with the further restriction that t shall be a multiple of α. This group induces on the marks of the field a doubly transitive group of degree p^n and order $p^n(p^n-1)n/\alpha$. When $\alpha = n$ this is the sole doubly transitive group of degree p^n and order $p^n(p^n-1)$ whose regular subgroups of order p^n-1 are cyclic.

By way of digression it may be pointed out that if we adjoin to the generators (7.2) the transformation $x' = 1/x$ then we are led to a group of order

$$(p^n+1)p^n(p^n-1)n/\alpha$$

which permutes ∞ and the marks of $GF[p^n]$ according to a triply transitive group of degree p^n+1. When p is odd each of these groups contains transformations whose determinants are not squares; then the elements whose determinants are squares constitute a subgroup of index 2 which is doubly transitive on the symbols involved.

We shall now determine all the doubly transitive groups G of degree p^n and order $p^n(p^n-1)$ contained in the group induced by the $p^n(p^n-1)n$ transformations (7.1). Such a group contains a regular permutation group M of degree and order p^n-1 on the non-zero marks of the field. There is one and just one group G in which the corresponding group M is cyclic; it is generated by the first two elements in (7.2). Henceforth let M be non-cyclic. It is obvious that the transformation group T, by which M is induced, consists of transformations of the form

$$(7.3) \qquad x' = a_i x^{p^{t_i}}, \qquad (i = 1, 2, \cdots, p^n-1, \quad 0 \leqq t_i < n).$$

Such a transformation replaces the mark $x = 1$ by the mark $x' = a_i$. Since M is regular on the non-zero marks of $GF[p^n]$ it follows that the coefficients a_i are in some order the non-zero marks of the field (without repetition or omission).

Now the totality of linear transformations in T constitutes a subgroup

of T; and this subgroup is contained in the cyclic group generated by the transformation S,

$$S: \qquad x' = \omega x,$$

where ω is a primitive mark of $GF[p^n]$. Then there exists a least positive integer σ such that this linear subgroup is generated by S^σ. It is clear that σ is a factor of the order $p^n - 1$ of S. If $\sigma = 1$ we have a cyclic group T, a case which we have already excluded; therefore, $\sigma > 1$. Then some of the exponents t_i are positive.

Let t be the least positive value of t_i appearing in the transformations (7. 3) of T; and let the transformation U,

$$U: \qquad x' = ax^{p^t},$$

be one of the transformations in which $t_i = t$. By taking successive powers of U we obtain transformations with the exponents $t, 2t, 3t, \cdots$ on p. Since these are to be reduced modulo n (on account of the equation $u^{p^n} = u^{p^0}$ for marks of $GF[p^n]$) it follows that t is a factor of n. Moreover, since t is the least positive value of an exponent t_i, each t_i must be a multiple of t; whence one concludes that the exponents t_i are $t, 2t, 3t, \cdots$. If T_1 and T_2 are two transformations in T with the same value of the exponent t_i, then $T_1^{-1}T_2$ is a linear transformation and hence is in $\{S^\sigma\}$. Therefore all the transformations in T having a given value of t_i are products of the form T_1S_1 where S_1 is in $\{S^\sigma\}$. Therefore T is generated by S^σ and U. The smallest positive value of λ such that U^λ is in $\{S^\sigma\}$ is $\lambda = n/t$. Since T and $\{S^\sigma\}$ are of orders $p^n - 1$ and $(p^n - 1)/\sigma$ it follows that $\sigma = n/t$ and hence that σ is a factor of n.

We have now to determine the further conditions on σ, t and a such that the group $\{S^\sigma, U\}$ shall indeed induce a permutation group of the type prescribed for M. If d is the greatest divisor of σ such that a is a d-th power of a mark in $GF[p^n]$, then every coefficient in the transformations belonging to $\{S^\sigma, U\}$ is a d-th power. Since d is a factor of $p^n - 1$ and every mark of $GF[p^n]$ occurs among these coefficients it follows that $d = 1$. Therefore if γ is such that $a = \omega^\gamma$ we must have γ prime to σ. We may now combine the transformation U with an appropriate power of S^σ so that in the resulting transformation $U_{l,t}$ of the form U (with the same value of t) we shall have the corresponding coefficient of the form ω^l where $0 < l < \sigma$ and l is prime to σ. Then we have

$$U_{l,t}: \qquad x' = \omega^l x^{p^t}, \quad 0 < l < \sigma, \quad l \text{ prime to } \sigma, \quad \sigma t = n.$$

Then $\{S^\sigma, U\} \equiv \{S^\sigma, U_{l,t}\}$.

The λ-th power of $U_{l,t}$ may be written in the form

$$U^{\lambda}_{l,t}: \qquad x' = \omega^{l(1+p^t+p^{2t}+\cdots+p^{(\lambda-1)t})} x^{p^t}.$$

The least positive value of λ for which this is in $\{S^\sigma\}$ is $\lambda = n/t = \sigma$. In order that the induced permutation group M shall be regular it is further necessary that the least value of λ for which

$$1 + p^t + p^{2t} + \cdots + p^{(\lambda-1)t}$$

shall be a multiple of σ is $\lambda = \sigma$, since otherwise at least one mark of $GF[p^n]$ would occur as a coefficient in two transformations belonging to $\{S^\sigma, U_{l,t}\}$.

When the necessary conditions now obtained are satisfied we shall easily show that $\{S^\sigma, U_{l,t}\}$ permutes the non-zero marks of $GF[p^n]$ according to a regular permutation group M. The coefficients in the transformations belonging to $\{S^\sigma, U_{l,t}\}$ are the marks

$$\omega^{k\sigma} \cdot \omega^{l(1+p^t+p^{2t}+\cdots+p^{(\lambda-1)t})}, \quad [k = 1, 2, \cdots, (p^n - 1)/\sigma, \lambda = 1, 2, \cdots, \sigma - 1],$$

together with the σ-th power marks appearing as coefficients in the transformations of $\{S^\sigma\}$. No two of these coefficients are equal since the second exponent on ω in the foregoing expressions is not a multiple of σ and no two such exponents have their difference a multiple of σ. Therefore no two transformations in $\{S^\sigma, U_{l,t}\}$ have the same coefficient and hence that group replaces the value of 1 of x by every non-zero mark of the field; whence it follows that $\{S^\sigma, U_{l,t}\}$ induces a regular permutation group M on the non-zero marks of the field.

Since $\sigma > 1$ it is easy to verify that the group $\{S^\sigma, U_{l,t}\}$ is non-Abelian; for the equation $S^{-\sigma} U_{l,t} S^\sigma = U_{l,t}$ would imply that $\sigma\,(p^t - 1) \equiv 0 \bmod p^n - 1$, and this is impossible since

$$\sigma(p^t - 1) < (p^\sigma - 1)(p^t - 1) = (p^{n/t} - 1)(p^t - 1) < p^n - 1 \quad \text{when} \quad \sigma < n.$$

Among the results established by the foregoing argument we have the following theorem (and thence the easily established corollaries):

THEOREM. *Every non-cyclic group T which is contained in the group whose elements are the transformations*

$$x' = ax^{p^t} + b, \qquad (a \neq 0, \quad t = 0, 1, \cdots, n - 1),$$

where a and b are marks of $GF[p^n]$, subject to the condition that T shall be of order $p^n - 1$ and shall permute according to a regular permutation group M the non-zero marks of $GF[p^n]$, is a non-Abelian group $\{S^\sigma, U_{l,t}\}$

where σ ($\sigma > 1$) is a common factor of n and $p^n - 1$ such that $\lambda = \sigma$ is the least value of λ for which $1 + p^t + p^{2t} + \cdots + p^{(\lambda-1)t}$ is divisible by σ, where $t = n/\sigma$; and every such group $\{S^\sigma, U_{1,t}\}$ is such a group T.

COROLLARY I. *If the elements of $\{S^\sigma, U_{1,t}\}$ are the transformations*

$$x' = a_i p^{t_i}, \qquad (i = 1, 2, \cdots, p^n - 1),$$

then the transformations

$$x' = a_i p^{t_i} + b_i, \qquad (i = 1, 2, \cdots, p^n - 1),$$

where for each value of i the symbol b_i runs over all the marks of $GF[p^n]$, induce a doubly transitive group of degree p^n and order $p^n(p^n - 1)$ on the marks of $GF[p^n]$ in which M is the largest subgroup each element of which leaves zero fixed.

COROLLARY II. *If $n = \sigma t$, $\sigma > 1$, and p is a prime of the form $\sigma z + 1$ then there exists a doubly transitive group of degree p^n and order $p^n(p^n - 1)$ whose regular subgroups of order $p^n - 1$ are non-Abelian.*

COROLLARY III. *Whenever n and $p^n - 1$ are not relatively prime there exist * at least two doubly transitive groups of degree p^n and order $p^n(p^n - 1)$.*

From the last corollary it follows that there are at least two distinct doubly transitive groups of degree p^2 and order $p^2(p^2 - 1)$ for every odd prime p. In no case does the theorem assert the existence of more than two such groups when $n = 2$. When $p^n = 3^2$ there are just two such groups. But when $p^n = 5^2$ or $p^n = 7^2$ there are three (and just three) such doubly transitive groups.

By aid of the foregoing theorem one may establish the following three theorems which we state without proof:

When p is an odd prime and n is an even integer there exist two triply transitive groups of degree $p^n + 1$ and order $(p^n + 1)p^n(p^n - 1)$. In one of these the regular subgroups of degree and order $p^n - 1$ are cyclic; in the other these subgroups are non-Abelian and contain cyclic subgroups of index 2.

The only triply transitive groups of degree $p^n + 1$ and order $(p^n + 1)p^n \times (p^n - 1)$ contained as subgroups in the triply transitive group of order

* This is in contradiction with a conjecture of Burnside [*Messenger of Mathematics*, Vol. 25 (1896), pp. 147-153; see also the footnote on p. 184 of the second edition of his *Theory of Groups*] to the effect that, with an exception in the case when $p^n = 3^2$, there is always one and just one doubly transitive group of degree p^n and order $p^n(p^n - 1)$, and in particular with the cases $n = 2$ and $n = 3$ in which he offered a supposed proof of the incorrect conclusion.

$(p^n+1)p^n(p^n-1)n$, described earlier in this section, are (1) one (a well known case) in which the regular subgroup of order p^n-1 is cyclic (existent for every p^n) and (2) the additional group described in the foregoing paragraph for the case when p is odd and n is even.

For every positive integer L three exists a prime p and a positive integer n such that the number of doubly transitive groups of degree p^n and order $p^n(p^n-1)$ is greater than L.

8. *The Algebras* $\boldsymbol{A}_{\sigma,l}[p^n]$. In the main theorem of the preceding section and its first corollary we have a means of defining an important class of algebras $A[p^n]$. We denote their elements by (a) where a runs over the marks of $GF[p^n]$ and where (0) and (1) are to be the zero and unit elements of the algebra respectively. Addition is defined by the relation $(a)+(b)=(a+b)$. For the product $(a_i)(x)$ we take the element $(a_i x^{p^{t_i}})$, where the symbols are those of the theorem cited and its first corollary. Such an algebra will be called an algebra $\boldsymbol{A}_{\sigma,l}[p^n]$, where σ and l are defined as in the theorem cited. They do not include all the algebras $A[p^n]$ as is shown by an examination of the three algebras $A[5^2]$ or the three algebras $A[7^2]$.

To construct three algebras $A[5^2]$ we proceed as follows. In the first place we have $GF[5^2]$ as one of the algebras. A second one is $A_{2,1}[5^2]$. To construct a third algebra $A[5^2]$ we observe that the transformations

$$x'=x+1, \qquad x'=\omega^8 x, \qquad x'=\omega^{12}x^5+\omega^{21}x,$$

where ω is a primitive mark of $GF[5^2]$ satisfying the relation $\omega^2=\omega+3$, permute the marks of this Galois field according to a doubly transitive group of degree 25 and order $25\cdot 24$. If an $A[5^2]$ is formed from this group by the method of § 1 it will be different from the other two $A[5^2]$ already described in this paragraph. It may be shown that the three $A[5^2]$ thus exhibited are all the possible algebras $A[5^2]$.

It seems probable that the algebras $\boldsymbol{A}_{\sigma,l}[p^n]$ are contained among the algebras otherwise constructed by Dickson (*l. c.*); but I did not seek to verify this proposition.

From the main theorem of the preceding section it follows that σ is a factor of

$$1+p^t+p^{2t}+\cdots+p^{(\sigma-1)t}=(p^n-1)/(p^t-1)$$

and hence it is a factor of

$$\frac{p^n-1}{p^t-1}\cdot\frac{p^t-1}{p-1}=\frac{p^n-1}{p-1}.$$

Therefore the order of the group $\{S^\sigma\}$ is a multiple of $p-1$ and hence that

group contains a cyclic subgroup of order $p-1$ and therefore contains all the transformations of the form $x' = \alpha x$ where α is an integral mark of $GF[p^n]$. Therefore in $\boldsymbol{A}_{\sigma,\iota}[p^n]$ we have $(\alpha)(x) = (\alpha x)$ where (x) is any element of the algebra and (α) is an integral element. But $(a_i)(\alpha) = (a_i \alpha) = (\alpha a_i)$ since $\alpha^p = \alpha$. Therefore an integral element of $\boldsymbol{A}_{\sigma,\iota}[p^n]$ is permutable under multiplication with every element of the algebra.

That this property of permutability of integral elements with all elements is not common to all algebras $A[p^n]$ may be seen from an examination of the last $A[5^2]$ defined earlier in this section and in fact from a consideration of the elements $x' = \omega^{12}x^5 + \omega^{21}x$ and $x' = \omega^{16}x$ in the underlying group.

Let k be any factor of n, and consider the subset (r) of elements in an algebra $\boldsymbol{A}_{\sigma,\iota}[p^n]$ where r runs over the marks of the subfield $GF[p^k]$ contained in $GF[p^n]$. Under addition these elements (r) obviously form a group of order p^k. Moreover, the product of two of these elements (r) is an element of this set. Therefore the p^k elements named form a sub-algebra $A[p^k]$. From this it follows readily that $\boldsymbol{A}_{\sigma,\iota}[p^n]$ contains a sub-algebra $A[p^k]$ when and only when k is a factor of n.

By means of the algebras $\boldsymbol{A}_{\sigma,\iota}[p^n]$ one may easily prove the following proposition: For every positive integer L there exists a prime p and a positive integer n such that the number of algebras $A[p^n]$ is greater than L.

CAYLEY DIAGRAMS ON THE ANCHOR RING.

By R. P. Baker.

1. Maschke * determined the Cayley color groups representable in the plane, postulating independence of generators. These are all half regular in Archimedes' sense. (There are the same number of polygons of the same orders at each vertex). Adding this as a postulated property this paper extends the enumeration to connectivities two and three.

2. On account of the importance in analysis of the 'groups of genus p,' I add them to the lists. Given classically with 'schraffirte" diagrams there is a corresponding Cayley diagram the method of construction being indicated by the plate in Burnside's Theory of Groups, Ch. XIX. The generators of these groups are not independent but connected by a relation of the form $ABC \cdots K = 1$. They do not therefore occur in Maschke's list. Adding them and two 'general' Cayley diagrams for the groups of order four on the tetrahedron it appears that all the half regular and regular bodies except two are the basis for Cayley diagrams.

In the list the net is specified by the orders of the polygons at a vertex.

	Net	Order	Generators	Group	Maschke's figure
1.	3.3.3	4	2.2.2	G_4	(general)
		4	4.2	C_4	(general)
2.	4.4.4	8	4.2	$G_8{}^4$	2
			4.2	Abelian	2a
			2.2.2	Abelian	16
3.	3.3.3.3	6	3.2.2	G_6	$(p=0)$
4.	3.3.3.3.3	12	3.3.2	$G_{12}{}^4$	$(p=0)$
5.	5.5.5	20		None	
I	3.6.6	12	3.2	$G_{12}{}^4$	3
II	3.8.8	24	3.2	$G_{24}{}^4$	6
II				$G_{12}{}^4C_2$	6a
III	3.10.10	60	3.2	$G_{60}{}^5$	9
IV	4.6.6	24	4.2	$G_{24}{}^4$	5
			2.2.2	$G_{24}{}^4$	17
V	4.4.n	2n	n.2	Dihedral	2
			2.2.2	Dihedral	16
VI	5.6.6	60	5.2	$G_{60}{}^5$	8
VII	3.4.4.4	24	4.3	$G_{24}{}^4$	7
VIII	3.4.3.4	12	3.3	$G_{12}{}^4$	4
IX	3.5.3.5	30		None	

* *American Journal of Mathematics*, Vol. 18 (1896), p. 156.

	Net	Order	Generators	Group	Maschke's figure
X	3.3.3.n	2n	n.2.2	Dihedral	$(p=0)$
XI	3.3.3.3.4	24	2.3.4	Octahedral	$(p=0)$
XII	3.3.3.3.5	60	2.3.5	Icosahedral	$(p=0)$
XIII	4.6.8	48	2.2.2	Extended octahedral	18
XIV	4.6.10	120	2.2.2	Extended icosahedral	
XV	3.4.3.4.5	60	3.5	$G_{60}{}^5$	10

3. Extending the problem to higher connectivities we are confronted by an embarassment of riches.

$G_{24}{}^4$ with three generators of order two has a half regular representation on the anchor ring and 72 other representations, mostly bizarre.

The Abelian G_9 $(3, 3)$ has representations not half regular, on the projective plane and on the anchor ring.

Taking the elements as $(1, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta, \beta^2, \alpha\beta^2, \alpha^2\beta^2)$ and numbering from $1 \cdots 9$ the polygons are:

(125697); (1364), (4587), (2398); (123), (456), (789), (147), (258), (369).

and in the second case,

(145693), (458936); (2365), (4786), (1287); (123), (147), (258), (896).

The diagrams are not perspicuous and lack the symmetry which is the essential characteristic of a group.

These facts together with the number of half regular diagrams discovered here justify the restricting hypothesis that the net is half regular.

4. For half regular nets the extended Euler equation becomes

$$\Sigma(l-2)/2l = 1 + (k-3)/V$$

where the l's are the orders of the polygons at each vertex, k is the connectivity and V the number of vertices. This is a necessary but not a sufficient condition. A sufficient condition is I believe not known, but we may note two types of failure. $l_1 = 5$, $l_2 = 5$, $l_3 = 10$, $k = 3$ satisfy the condition but no net can be constructed. The statement does not allow us to bound a single pentagon. This may be called failure 'im kleinen.'

$l = 3, 4, 4$ $V = 6$, $k = 1$ belongs to the triangular prism but $l = 3, 4, 4$ $V = 3$, $k = 2$ also satisfies the equation and there is no such figure in the projective plane. This may be called a failure 'im grössen.'

If a solution $(l_1, l_2, \cdots l_n)$ k, V exists and also a solution $(l_1, l_2, \cdots l_n)$ k', V' which is possible if $(k-3)V' = (k'-3)V$ then if V' is a multiple of V and the first solution has a net the second has, but not conversely. For $k = 2$, $k' = 1$ the construction can be carried out by Klein's

double representation of the projective plane and joining up the two nets. In the higher cases we must combine this method with that of connecting overlying Riemann surfaces by branch cuts following Dyck.*

5. The enumeration of the diagrams possible in the projective plane can be made speedily as we have the list for $k = 1$ and can pick out those which can be bisected by a simple closed curve not passing through a vertex. One condition for bisection is that it must be possible to arrange the points on the sphere so as to exhibit central symmetry. This is met in all cases. A further condition is that the opposite edges must be of like color. No. 17 fails here. Still further the arrows must not conflict with the reversal of the Möbius indicatrix. This means that the arrows if any must be alternately clock and counterclock. This cuts out Nos. 3.4.5.6.7.8.9.10. In No. 2, the even prisms lose the independence of generators on bisection.

No. 6a gives G_{12}^4 and the three color group 18 gives G_{24}^4. The extended icosahedron group (not drawn in Maschke's paper) G_{60}^5 with three generators of order two.

These three groups are the subjects of extension in the extended groups of Maschke's list.

Cayley Diagrams in the Projective Plane (Plate I).
(Half regular).

Net	Order	Generators	Group
3.8.8	12	3.2	G_{12}^4
4.6.8	24	2.2.2	G_{24}^4
4.6.10	60	2.2.2	G_{60}^5

6. For the anchor ring, by dissection and development to a parallelogram, and then by indefinite repetition in the plane, the problem is reduced to one of homogeneous assemblages in the plane. If we consider such a development colored and arrowed on one of the nets and being a Cayley diagram for a group, certain of the relations can be determined by inspection of any small region. Such are the order of the generators and the definition of the intermediate polygons. These we may call 'im kleinen' while the others which demand a knowledge of the 'cut' that is of the fundamental region are properly called 'im grössen' relations.

The latter will in general contain one or more arbitrary integers due to the fact that V disappears from the Euler equation or, geometrically ex-

* *Mathematische Annalen*, Vol. 17 (1880), p. 473.

pressed, to the arbitrary nature of the fundamental region. The heuristic program is then to determine (1) all possible half regular nets (2) all consistent colorings for each net (3) all consistent arrowings.

Then for each case we have the 'im kleinen' relations and can discuss the possible 'im grössen.'

There are some cases where the 'im kleinen' determine the 'im grössen' and one or more integers disappear from the expression for the order. In the more general case it is possible to classify the points into sets invariant under a subgroup called by Burnside, in his discussion of groups of genus one, the group of translations. The word is hardly well chosen here for often some elements are moved in opposite senses a phenomenon which can be associated with the idea of an electrolytic vector, but more complicated cases exist where generalized 'ballet' seems needed.

Falling back on the technical language of group theory this subgroup is a maximal self conjugate abelian subgroup and the order of the quotient group gives the number of classes of points.

7. *The half regular nets on the anchor ring.* The equation

$$\sum_{i=1}^{i=n} (l_i - 2)/2l_i = 1$$

has 17 solutions.

$n = 3$ (6, 6, 6), (12, 12, 3), (8, 8, 4), (10, 5, 5), (42, 7, 3), (24, 8, 3), (18, 9, 3), (15, 10, 3), (20, 5, 4), (12, 6, 4),
$n = 4$ (4, 4, 4, 4), (6, 3, 6, 3), (6, 4, 3, 4), (12, 4, 3, 3),
$n = 5$ (4, 4, 3, 3, 3), (6, 3, 3, 3, 3),
$n = 6$ (3, 3, 3, 3, 3, 3).

We note first that any solution with two odd numbers and one even fails 'im kleinen': we cannot properly bound an odd polygon. Similarly two different evens and one odd fails to properly bound the odd. The case (12, 4, 3, 3) leads eventually to either two twelves or two fours at a vertex.

This leaves ten cases:

(6, 6, 6), (12, 12, 3), (8, 8, 4), (12, 6, 4), (4, 4, 4, 4)
(6, 3, 6, 3), (6, 4, 3, 4), (4, 3, 4, 3, 3), (3, 3, 3, 3, 3, 3).

For those with 4 or 5 polygons the cyclic order given is the only possible.

The duals of these nets give all the 'space fillers' for the plane. Those with all evens are the basis for the 'schraffirte' diagrams of the groups of genus one.

Metric solutions can be constructed in several ways employing only the draftsman's triangles and one opening of the compasses.

8. Before proceeding to the detailed enumeration of cases certain general features of the configurations may profitably be discussed, and an explanation given of the tabular entry of the final results. Going from the table to the picture, the net and the 'im kleinen' relations enable us to construct the homogeneous assemblage colored and arrowed. I use throughout the following symbols for the generators and colors

α	————————	blue
β	— · — · — · — · — ·	red
γ	· · · · · · · · · · · ·	green
δ	· · · — · · · — · · · —	yellow

In the table 'con.' means that the arrows are all of the same sense, while 'alt.' implies that half of them are of each sense. In one case (41) reference to the picture shows the exact sense, there being two possible alternate arrangements of which only one is possible for us. In 5 and 5′ two arrangements are possible. The choice of generators X, Y for the abelian subgroup is necessarily arbitrary. It is always possible to arrange a fundamental region as a parallelogram (or rhombus if X and Y are conjugate) with the identity at each vertex. These cases are listed as (a). There are other possibilities however. If we take X for some point as leading to the right and of order k and Y leading downwards from the same point and the t-th row down parallel to the X line as the first row with the identity, at the point r steps from the Y line so that $Y^tX^r = 1$ there will in general be relations between k, t and r. Figure 1 shows a colored net with the net of X, Y, superimposed.

To determine these we consider points not on the XY net. Under the operations X, Y these points in general move in different directions but since each aggregate of points one of each class which neighbor or form a molecule must be gathered together again at the identity there are other relations of the type $Y^\tau X^\rho = 1$ but not always identical with the first.

The totality of these relations, not usually independent, must be satisfied and restrict the arithmetic nature of the order and the structure of the group.

One special case may be reported in full. The net 8.8.4 may be colored with the squares of one color α and of one sense and the other lines with a second color β. This gives the 'im kleinen' relations $\alpha^4 = \beta^2 = (\alpha\beta)^4 = 1$ and X, Y may be taken as $(\alpha^2\beta)$ and $(\alpha\beta\alpha)$ respectively for these operators carry into themselves points of each class and are commutative and generate a self conjugate subgroup. Moreover it is a maximal group of this kind for all other operations carry points of one class to points of another.

X and Y being conjugate we have $X^k = Y^k = 1$. The 'molecule' may

be the four points of a quadrilateral and as drawn X carries (1) to the right (2) up, (3) left and (4) down. Y carries (1) down (2) right (3) up and (4) left. The relation $Y^tX^r = 1$ for (1) becomes for (2) $X^tY^{-r} = 1$ and for (3) $Y^{-t}X^{-r} = 1$ and for (4) $X^{-t}Y^r = 1$. Of these the first two are independent giving $X^{r^2+t^2} = Y^{r^2+t^2} = 1$ and $r^2 + t^2$ must contain k as a factor. Now if r and t are relatively prime we may determine p and q so that

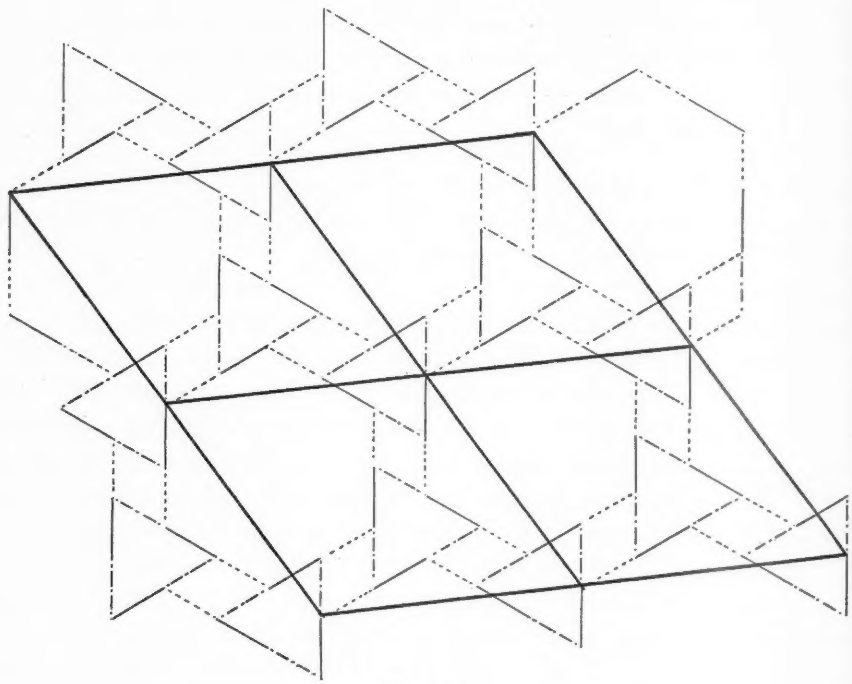


FIG. 1

$YX^s = 1$ which means that the identity occurs in the first row down amending our first statement. $s = pr - qt$.

If r and t have a common divisor m we obtain similarly a relation of the form $Y^mX^{mr} = 1$, which is taken as a standard form for representations of type b. In this case the order of the abelian subgroup is $m^2(r^2 + 1)$ and of the whole group $4m^2(r^2 + 1)$, the quotient group being C_4. This is determined in the usual manner by setting up a table for the group with $[X, Y]$ in the first line.

The groups whose numbers are given with a * are groups of genus one

with one generator removed. The order is given in this reduced form and differs apparently from Burnside's order but is arithmetically equivalent. The reduction is essential for the draftsman. The groups then occur in sets whose orders are square multiples of a primitive $m=1$ and whose developments are geometrical repetitions of this in square array. The two forms given by Dyck (*loc. cit.*) are special forms for $r=0$ and $r=1$ (or 2). Since the form (a) cannot in every case be obtained from the form (b) by placing $r=0$ I list them separately in all cases.

In the case of groups of genus one of class I Burnside gives the order as $2(ab'-a'b)$ with $(ab'-a'b) \neq 0$ for my postulates this must be amended by saying $(ab'-a'b)$ is not a prime for in this case a second generator loses its independence.

As a final check on the correctness of the pictures Maschke's theorem (2) has been used. This is equivalent to demanding that if the points be numbered and the operators expressed as substitutions the group generated is regular.

9. The picture of the development being established the next proceeding is to place it on the anchor ring. This is arbitrary in several ways. In the first place X and Y may run round either of two non-homologous circuits and may start at a point of any class. Secondly the whole picture is subject to continuous deformation.

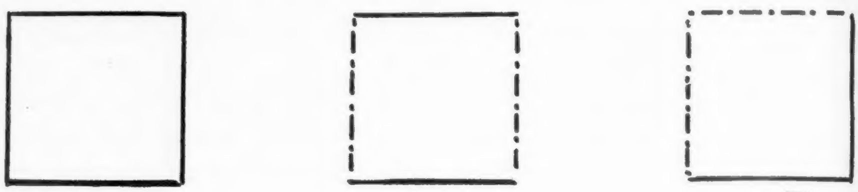
There is at least one way of fixing the position of the points on mathematical principles. If we accept the drawing of the net as standard and the proportions of the anchor ring are agreed on the fundamental parallelogram may be treated as belonging to an elliptic function, transferred to a two-sheeted Riemann surface and thence conformally to the anchor ring. The required calculations are obviously rather severe and the result might be aesthetically more satisfactory and might not. Maschke did not use the conformal stereographic projection and my experiments with that method confirm his judgment. I have only tried to attain perspicuity.

In the drawings the left hand figure can be considered at the front of the anchor ring seen from the outside and the right hand one as the back seen from the inside so that homothetic points on the edges agree. This causes an apparent reversal of arrows on homothetic cycles which cut out 'flachenstücken' but not of cycles encircling the hole. This method seems psychologically preferable to the plan adopted with coins.

10. *Proceeding to the detailed enumeration.*

The net (4.4.4.4).

The only possible colorings for a square are I all α's, II alternate α and β, III two adjacent α's and two adjacent β's.



I $\alpha\alpha$ II $\alpha\beta$ III $\overline{\alpha\beta}$

FIG. 2

For the discussion of the possible sets of four squares at a corner these types may be denoted by $\alpha\alpha$, $\alpha\beta$, $\overline{\alpha\beta}$ respectively.

If there are four colors $\alpha\alpha$ and $\overline{\alpha\beta}$ are impossible and four squares at a corner must be of types $(\alpha\beta \mid \alpha\gamma)/(\beta\delta \mid \gamma\delta)$, so that the horizontal lines read $\beta\gamma\beta\gamma\beta\gamma\cdots$ and the vertical $\alpha\delta\alpha\delta\cdots$. The operators necessarily of period two are by this coloring compelled to have α, δ each permutable with β and γ. $(\alpha\delta)$ and $(\beta\gamma)$ are not necessarily conjugate and for the type (a) the fundamental region is a $2k \times 2l$ rectangle with $(\alpha\delta)^k = (\beta\gamma)^l = 1$ as the 'im grössen' relations. For the type (b) the (t, r) technique gives $Y^tX^r = Y^{-t}X^r = 1$ whence $X^{2r} = Y^{2t} = 1$ and the identity occurs at two corners and the midpoint of the opposite side of a $2r \times t$ rectangle. There are four classes of points, a molecule may be taken as the four points on any square while the quotient group works out as G_4. If the independence of operators is to be kept k, l must be greater than unity. In type (b) there are no restrictions on t, r. A possible amendment to the general forms in the cases where the integers entering take the values 1 or 2 is left to the reader. These two cases are listed as 1(a) and 1(b).

If there are three colors.

The squares of type $\overline{\alpha\beta}$ cannot be used, they demand two colors. Taking first the case where all the squares are $\alpha\beta$, the four at a corner must be equivalent to $(\alpha\beta \mid \alpha\beta)/(\alpha\gamma \mid \alpha\gamma)$. The α color must run through. This gives the 'im kleinen' relations $\alpha^k = \beta^2 = \gamma^2 = 1$.

If the arrows concur α permutes with $(\beta\gamma)$ and these can be used as X and Y. The (t, r) argument yields the same result as in 1(b) and the two cases are listed as 2(a), 2(b).

If the arrows alternate the situation is different, a general r is possible, the cases are 3(a), 3(b). In both (2) and (3) the quotient group is C_2.

If with three colors any square is of type $\alpha\alpha$ the arrangement at a corner is $(\alpha\alpha \mid \alpha\beta)/(\alpha\gamma \mid \beta\gamma)$ which is a self perpetuating system.

If the arrows are all counterclock an internal square reads $\alpha\beta\alpha\beta = 1$ which is also the reading of an edge. The group is of order 16, the 'im kleinen' relations determining the 'im grössen.' This occurs also if the arrows alternate. The cases (4) and (5) exhibit G_{16} III vii (Burnside's enumeration) and the abelian (4, 2, 2).

If there are two colors:

A square of the $\alpha\alpha$ type must meet a $\beta\beta$ square at each end of one of its diagonals and at each corner there is the arrangement $(\alpha\alpha \mid \alpha\beta)/(\alpha\beta \mid \beta\beta)$ which is self perpetuating.

If each set of arrows is counterclock the 'im kleinen' relations are $\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$ and $(\alpha^3\beta)$ and $(\alpha\beta^3)$ serve as $X, Y \ldots$

The (t, r) method gives the common order of these as $m(r^2 + 1)$ provided $r \neq 0$ and for $r = 0$ the (b) case includes the (a). For graphical distinctions however I list two cases 6(a) and 6(b). These are groups of genus one.

If the arrows of each set alternate there arises the abelian group of order 16 (4, 4), while if one set concur and the other alternates the group G_{16} III viii the 'im kleinen' dominating (7) and (8).

With two colors and all squares of the second type the α and β lines intersect, a self perpetuating situation. The general case has a group of order mtr without restriction save $m, t, r > 1$. In the special case the order is mn. Both groups are abelian the quotient group being the identity. 9(a), 9(b).

If the arrows of one set concur and the others alternate the groups are of doubled order but the rest is similar. 10(a), 10(b).

If both sets of arrows alternate the quotient group is G_4 the rest of the situation similar with groups of fourfold order. 11(a), 11(b).

All the squares may be of the third type $\overline{\alpha\beta}$, the color lines being corrugated. With concurrent arrows there are two cases 12(a), 12(b).

If the arrows concur for one set and alternate for the other we have both $\alpha^2 = \beta^2$ and $\alpha^2 = \beta^{-2}$. The group is the quaternion group. (13).

If both sets alternate we have the abelian group [4, 2]. (14).

If some squares are of type III but not all, the others must be of type II. A square of type III implies a whole diagonal row of the same type and if a square of type II joins it there must be a whole diagonal row of the same next the row of III's.

Taking first the case of concurrent arrows in both sets and considering two columns the III square gives $\alpha^2=\beta^2$ and the II square $\alpha\beta=\beta\alpha$. The identity recurs across the diagonal of the two III squares. It is then only necessary to consider these two columns. The results depend on the sequence of the types in the columns. This is the same for each and we can assume that it starts with a III.

Let this sequence be $(\mathrm{III})^{k_1}\,(\mathrm{II})^{l_1}\,(\mathrm{III})^{k_2}\,(\mathrm{II})^{l_2}\,(\mathrm{III})^{k_3}\,(\mathrm{II})^{l_3}\cdots$.

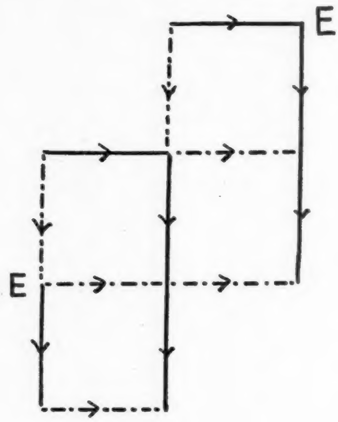


FIG. 3

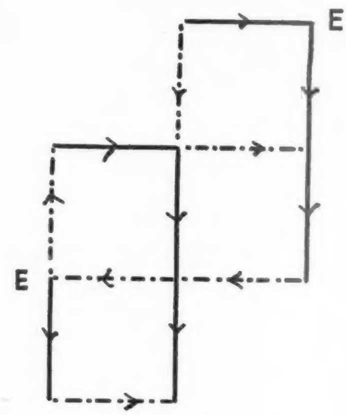


FIG. 4

The total number in the sequence must be $2k$ the order of α and β. Now the III's change the color on the right edge and the II's continue them so that if the number of III's is odd the last segment on either the right edge or the mid line reads $\alpha^{2k-1}\alpha^{2k}$ and the configuration is non-oriented. If the number of III's is even there are two cases.

If the number of β's is odd the identity does not recur till the second column is filled and the whole could be rearranged as a single column of squares. If the number of β's is even the identity recurs at the bottom of the right edge and there is a two column set (15a), (15b).

The number of β's is odd or even with $\Sigma(k_i/2)+\Sigma l_i+\Sigma'(k_i j_i)-\Sigma''(k_i l_j)$ where in Σ' $j>i$ and in Σ'' $j\geq i$.

If the arrows for α concur and those for β alternate one III square gives $\alpha^2=\beta^2$ and the other $\alpha^2\beta^2=1$ and the II squares $\beta\alpha\beta=\alpha$. The group is the quaternion group in two column form for the sequence III, II, III, II and in one column for III, II, II, III. (16), (17).

If the arrows both alternate the III squares read $\alpha^2=\beta^2$ and the II squares $(\alpha\beta)^2=1$. In this case one can prove $\alpha^8=1$ but not $\alpha^4=1$. There

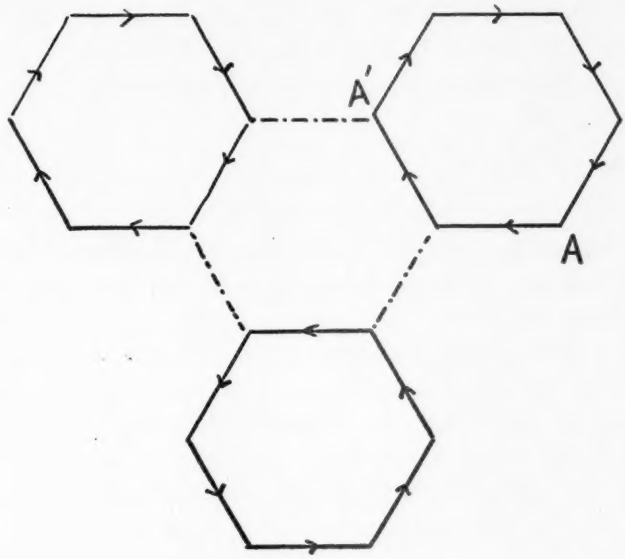


FIG. 5

No hexagon of type I can join one of type III or IV or V.

If I joins a VI it is surrounded by VI's and there is a cycle of 18 α's whereas $\alpha^6 = 1$.

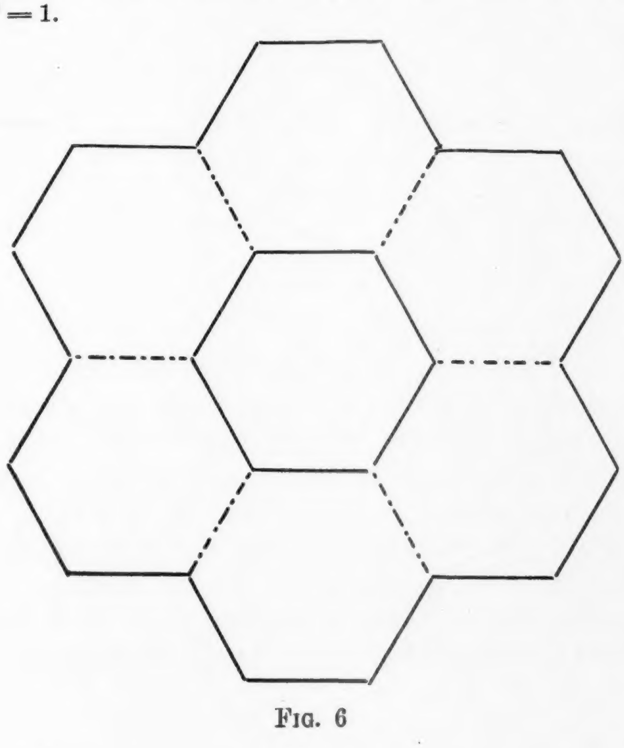


FIG. 6

is a group of order 16 with four columns or rearranged as a 4×4 square with cycles III, II, III, II in both columns and rows. (20).

If $\alpha^4 = 1$ there are two and one column representations of the abelian [4, 2]. (18), (19).

The net (6.6.6).

There are six types of hexagon available.

I α^6, II $(\alpha\beta)^3$, III $(\alpha\beta\gamma)^2$, IV $\alpha\beta\gamma\alpha\gamma\beta$, V $(\alpha^2\beta)^2$, VI $\alpha^3\beta\alpha\beta$.

Type I may be surrounded by type II and with concurrent arrows gives rise to two cases (25a) and (25b), groups of genus one.

If the arrows alternate there is a contradiction. Somewhere on the diagram, of the three I's neighboring a II two must be clock and the third counterclock. The II hexagon reads $\alpha\beta\alpha\beta\alpha^{-1}\beta = 1$ and starting from A with this operation we reach A′ with the contradiction $\alpha^2 = 1$ for $\alpha^6 = 1$.

If there are no I's but a II, this may be joined by II's of other pairs of colors with two kinds of fundamental region. (24a), (24b).

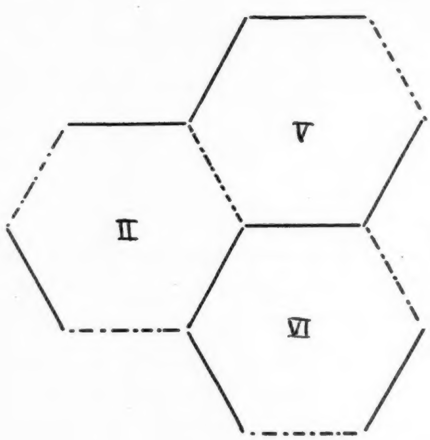


FIG. 7

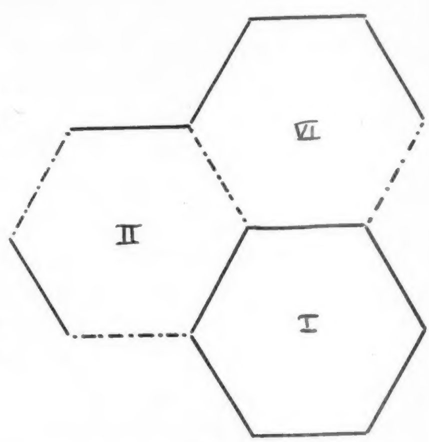


FIG. 8

A II cannot join a III. If a II join a IV it is surrounded by IV's and the outer cycle reads $(\alpha\beta)^9 = 1$, a contradiction.

If a II join a V on the β side it also joins a VI. The relations $(\alpha\beta)^3 = \alpha^2\beta\alpha^2\beta = \alpha^3\beta\alpha\beta = 1$ lead to $\beta = \alpha^2$.

If a II join a VI the common neighbor is a I and this has been disposed of.

Hexagons of type III can exist alone the X, Y not being conjugate. The groups are of genus one and graphically fall into two classes. 21a, 21b.

A III cannot join any one or two colored hexagon. III's and IV's can exist together in alternate rows and as $\beta\gamma$ is of order two there are only two rows. (23). Here the 'im kleinen' relations remove one of the parameters from the order but not both.

Hexagons of type IV can exist alone in two ways 22a, 22b.

Hexagons of type IV cannot be used with V or VI.

All the hexagons may be of type V and there are four cases with arrows concurrent or alternate and two kinds of fundamental region. 26a, 26b, 27a, 27b.

VI's can exist alone but the 'im kleinen' relations again control the size. With concurrent arrows the step from A to B is $\alpha^4\beta$ and also $\beta\alpha^4$. The hexagon gives $\alpha^3\beta\alpha\,\beta=1$ hence $\alpha^8=1$. There is a G_{16} (28) and a G_8 (29) with alternate arrows similarly a G_{16} (30) and G_8 (31). If V and VI both occur a similar argument leads to $\alpha^4=1$ for both ways of placing the arrows but the edge of the fundamental region shows that the figure is non-oriented.

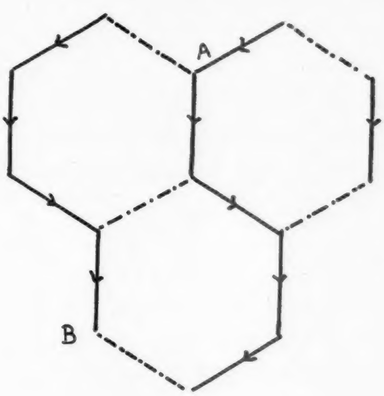


FIG. 9

The net (8.8.4).

With two colors. Squares of type III (see net 4.4.4.4) cannot occur as they demand vertices of the fourth order. The octagons cannot have an even succession of one color for one of the neighbors at the end of the run must be a square and cannot be either I or II. The permissible octagons are then reduced to the list:

$$\text{I } \alpha^8, \quad \text{II } \alpha^3\beta\alpha^3\beta, \quad \text{III } \alpha^3\beta\alpha\beta\alpha\beta, \quad \text{IV } \alpha\beta\alpha\beta\alpha\beta\alpha\beta, \quad \text{V } \alpha^5\beta\alpha\beta.$$

From the relations only, the only coexistent types are I, II with the squares reading $\alpha\beta\alpha\beta$; I, IV the squares reading $\alpha\beta=\beta\alpha$, and II, IV the squares being $\alpha\beta\alpha\beta$. The geometry prevents I and II adjoining. If I, IV join each

type is surrounded by the other and IV must read $(\alpha\beta\alpha^7\beta)^2=1$ while the squares give $\alpha\beta\alpha^7\beta=1$ a contradiction in the diagram if not in the abstract symbols. II and IV cannot join.

The octagons must then be all of one type. They cannot be all I's or all III's. V's can be arranged in a pattern but the octagon reads $\beta\alpha=\alpha^5\beta$ if the squares read $\alpha\beta=\beta\alpha$ and $\alpha^5\beta\alpha=\beta$ if the squares are $\alpha\beta\alpha\beta$. In each case $\alpha^4=1$ contradicts V.

If the octagons are all of type II and arrows concur we have (36) and if they alternate (37) with loss of a parameter in each case.

If the octagons are all IV's and arrows concur $\alpha^4=\beta^2=(\alpha\beta)^4=1$.

The groups are of genus one of order $4m^2(r^2+1)$, or with $r=0$. (32a) and (32b). If the arrows alternate the order is doubled (33a) and (33b).

With three colors. If α, γ occur in one square they occur in all. There are two possible arrangements, homothetic and alternate. (34a), (34b), (35a), (35b). These may be derived from (32) and 33) by replacing the α^4 squares by $\alpha\gamma\alpha\gamma$, a method used by Maschke.

The net (12.12.3).

The triangle must read $\alpha^3=1$, the twelve side either $(\alpha\beta)^6=1$ or $(\alpha\beta\alpha^2\beta)^2=1$ according to the arrows. The concurrent case gives groups of genus one and the alternate case has the same r function (r^2-r+1). (38a), (38b), (39a), (39b).

The net (6.3.6.3).

The triangles must read $\alpha^3=1$, $\beta^3=1$. With concurrent arrows the groups are of genus one. (40a), (40b). With alternate arrows and the particular arrangement of the picture there is a single group of order 24. (41).

The net (6.4.3.4).

The triangle must read $\beta^3=1$ and the hexagons $\alpha^6=1$ for $(\alpha\gamma)^3=1$ leads to a conflict on progressing round a triangle. Concurrent arrows give groups of genus one. Alternation is not possible.

The net (12.6.4).

No polygon can be of one color for the intermediate polygons conflict. With three colors there are two cases. 43a and 43b.

To the list for completeness are added the groups of genus one and the non-oriented groups.

Cayley Diagrams on the Anchor Ring.

Net (4.4.4.4).

No.	Order.	Im kleinen relations.	X	Y		Arrows.
1a	$4kl$	$\alpha^2=\beta^2=\gamma^2=\delta^2=1$	$(\alpha\delta)_k$	$(\beta\gamma)_l$	G_4	
1b	$8tr$	α, δ com with β, γ	$(\alpha\delta)_{2r}$	$(\beta\gamma)_{2t} Y^tX^r=1$	G_4	
2a	$2kl$	$\alpha^k=\beta^2=\gamma^2=1$ α com.	α_k	$(\beta\gamma)_l$	C_2	con.
2b	$4tr$	$\alpha^{2r}=\beta^2=\gamma^2=1$ α com.	α_{2r}	$(\beta\gamma)_{2t} Y^tX^r=1$	C_2	con.
3a	$2kl$	$\alpha^k=\beta^2=\gamma^2=(\alpha\beta)^2=(\alpha\gamma)^2=1$	α_k	$(\beta\gamma)_l$	C_2	alt.
3b	$2mtr$	$\alpha^{mr}=\beta^2=\gamma^2=(\alpha\beta)^2=(\alpha\gamma)^2=1$	α_{mr}	$(\beta\gamma)_{mt} Y^tX^r=1$	C_2	alt.
4	16	$\alpha^4=\beta^2=\gamma^2=(\alpha\beta)^2=(\alpha\gamma)^2=(\beta\gamma)^2=1$				con.
5	16	$\alpha^4=\beta^2=\gamma^2=(\alpha\beta)^2=1$ γ com.				alt.
5′	16	$\alpha^4=\beta^2=\gamma^2=1$ com.				alt.
6a*	$4k^2$	$\alpha^4=\beta^4=(\alpha\beta)^2=1$	$(\alpha^3\beta)_k$	$(\alpha\beta^3)_k$	C_4	con. con.
6b*	$4m^2(r^2+1)$	$\alpha^4=\beta^4=(\alpha\beta)^2=1$	$(\alpha\beta^3)_{m(r^2+1)}$	$(\alpha^3\beta)_{m(r^2+1)} Y^mX^{mr}=1$	C_4	con. con.
7	16	$\alpha^4=\beta^4=1$ com.				alt. alt.
8	16	$\alpha^4=\beta^4=1$ $\alpha\beta=\beta\alpha^3$				con. alt.
9a	mn	$\alpha^m=\beta^n=1$ com.	α_m	β_n	I	con. con.
9b	mtr	$\alpha^{mr}=\beta^{mt}=1$ com.	α_{mr}	β_{mt} $Y^tX^r=1$	I	con. con.
10a	$2mn$	$\alpha^{2m}=\beta^n=1$ $\beta\alpha\beta=\alpha$	$(\alpha^2)_m$	β_n	C_2	con. alt.
10b	$4tr$	$\alpha^{4r}=\beta^{2t}=1$ $\beta\alpha\beta=\alpha$	$(\alpha^2)_{2r}$	β_{2t} $Y^tX^r=1$	C_2	con. alt.
11a	$4mn$	$\alpha^{2m}=\beta^{2n}=(\alpha\beta)^2=1$	$(\alpha^2)_m$	$(\beta^2)_n$	G_4	alt. alt.
11b	$8tr$	$\alpha^{4r}=\beta^{4t}=1$ $\alpha^2=\beta^2$	$(\alpha^2)_{2r}$	$(\beta^2)_{2t}$ $Y^tX^r=1$	G_4	alt. alt.
12a	$2k^2$	$\alpha^{2k}=\beta^{2k}=1$ $\alpha^2=\beta^2$	$(\alpha\beta)_k$	$(\beta\alpha)_k$	C_2	con. con.
12b	$4k^2$	$\alpha^{4k}=\beta^{4k}=1$ $\alpha^2=\beta^2$	$(\alpha\beta)_{2k}$	$(\beta\alpha)_{2k}$ $Y^kX^k=1$	C_2	con. con.
13	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ $\beta\alpha\beta=\alpha$				con. alt.
14	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ com.				alt. alt.
15a	$4k$	$\alpha^{2k}=\beta^{2k}=1$ $\alpha^2=\beta^2$ com.			(2 columns)	con. con.
15b	$4k$	$\alpha^{2k}=\beta^{2k}=1$ $\alpha^2=\beta^2$ com.			(1 column)	con. con.
16	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ $\beta\alpha\beta=\alpha$			(2 columns)	con. alt.
17	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ $\beta\alpha\beta=\alpha$			(1 column)	con. alt.
18	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ com.			(2 columns)	alt. alt.
19	8	$\alpha^4=\beta^4=1$ $\alpha^2=\beta^2$ com.			(1 column)	alt. alt.
20	16	$\alpha^8=\beta^8=(\alpha\beta)^2=1$				alt. alt.

Net (6.6.6).

No.	Order.	Im kleinen relations.	X	Y		Arrows.	
21a*	$2kl$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma)^2=1$	$(\alpha\beta)_k$	$(\beta\gamma)_l$		C_2	
21b*	$2mtr$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma)^2=1$	$(\alpha\beta)_{mr}$	$(\beta\gamma)_{mt}$	$Y^tX^r=1$	C_2	
22a	$4kl$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma\alpha\gamma\beta)=1$	$[(\alpha\beta)^2]_k$	$(\beta\gamma)_l$		G_4	
22b	$8kl$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma\alpha\gamma\beta)=1$	$[(\alpha\beta)^2]_{2k}$	$(\beta\gamma)_l$	$X^k=Y^2$	G_4	
23	$16k$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma)^2=1$	$[(\alpha\beta)^2]_{2k}$	$(\beta\gamma)_2$		G_4	
24a	$6k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta)^3=(\beta\gamma)^3=(\alpha\gamma)^3=1$	$(\alpha\gamma\alpha\beta)_k$	$(\gamma\beta\gamma\alpha)_k$		G_6	
24b	$18k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta)^3=(\beta\gamma)^3=(\alpha\gamma)^3=1$	$(\alpha\gamma\alpha\beta)_{3k}$	$(\gamma\beta\gamma\alpha)_{3k}$	$X^k=Y^k$	G_6	
25a*	$6k^2$	$\alpha^6=\beta^2=(\alpha\beta)^3=1$	$(\alpha\beta\alpha^2)_k$	$(\alpha^2\beta\alpha)_k$		C_6	con.
25b*	$6m^2(r_{+}r^2+1)$	$\alpha^6=\beta^2=(\alpha\beta)^3=1$	$(\alpha\beta\alpha^2)_k$	$(\alpha^2\beta\alpha)_k$	$Y^mX^{mr}=1$; $k=m(r^2+r+1)$	C_6	con.
26a	$2kl$	$\alpha^{2k}=\beta^2=1$ $\alpha^2\beta=\beta\alpha^2$	$(\alpha^2)_k$	$(\beta\alpha)_l$		C_2	con.
26b	$4tr$	$\alpha^{4r}=\beta^2=1$ $\alpha^2\beta=\beta\alpha^2$	$(\alpha^2)_{2r}$	$(\beta\alpha)_{2t}$	$Y^tX^r=1$	C_2	con.
27a	$4kl$	$\alpha^{2k}=\beta^2=1$ $(\alpha^2\beta)^2=1$	$(\alpha^2)_k$	$(\beta\alpha\beta\alpha)_l$		G_4	alt.
27b	$8tr$	$\alpha^{4r}=\beta^2=(\alpha^2\beta)^2=1$	$(\alpha^2)_{2r}$	$(\beta\alpha\beta\alpha)_{2t}$	$Y^tX^r=1$	G_4	alt.
28	16	$\alpha^8=\beta^2=\alpha^3\beta\alpha^7\beta=1$					con.
29	8	$\alpha^4=\beta^2=\alpha^3\beta\alpha^3\beta=1$					con.
30	16	$\alpha^8=\beta^2=\alpha^3\beta\alpha\beta=1$					alt.
31	8	$\alpha^4=\beta^2=\alpha^3\beta\alpha\beta=1$					alt.

Net (8.8.4).

No.	Order.	Im kleinen relations.	X	Y		Arrows.	
32a*	$4k^2$	$\alpha^4=\beta^2=(\alpha\beta)^4=1$	$(\alpha^2\beta)_k$	$(\alpha\beta\alpha)_k$		C_4	con.
32b*	$4m^2(r^2+1)$	$\alpha^4=\beta^2=(\alpha\beta)^4=1$	$(\alpha^2\beta)_{m(r^2+1)}$	$(\alpha\beta\alpha)_{m(r^2+1)}$	$Y^mX^{mr}=1$	C_4	con.
33a	$8k^2$	$\alpha^4=\beta^2=(\alpha\beta\alpha^3\beta)^2=1$	$(\alpha^2\beta\alpha^2\beta)_k$	$[(\alpha\beta\alpha)^2]_k$		$G_8{}^4$	alt.
33b	$8m^2(r^2+1)$	$\alpha^4=\beta^2=(\alpha\beta\alpha^3\beta)^2=1$	$(\alpha^2\beta\alpha^2\beta)_{m(r^2+1)}$	$[(\alpha\beta\alpha)]_{m(r^2+1)}$	$Y^mX^{mr}=1$	$G_8{}^4$	
34a	$4k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma\beta)^2=1$	$(\alpha\gamma\beta)_k$	$(\gamma\beta\alpha)_k$		C_4	
34b	$8k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\beta\gamma\beta)^2=1$	$(\alpha\gamma\beta)_{2k}$	$(\gamma\beta\alpha)_{2k}$	$X^k=Y^k$	C_4	
35a	$4k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\gamma)^2=(\alpha\beta)^4=(\beta\gamma)^4=1$	$(\alpha\gamma\beta)_k$	$(\gamma\beta\alpha)_k$		G_4	
35b	$8k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\gamma)^2=(\alpha\beta)^4=(\beta\gamma)^4=1$	$(\alpha\gamma\beta)_{2k}$	$(\gamma\beta\alpha)_{2k}$	$X^k=Y^k$	G_4	
36	$8k$	$\alpha^{4k}=\beta^2=1$ com.	α_{4k}	β_2		I	con.
37	$8k$	$\alpha^{4k}=\beta^2=(\alpha\beta)^2=1$	α_{4k}			C_2	alt.

Net (12.12.3).

No.	Order.	Im kleinen relations.	X	Y	Arrows.	
38a*	$6k^2$	$\alpha^3=\beta^2=(\alpha\beta)^6=1$	$(\alpha^2\beta\alpha\beta)_k$	$(\alpha\beta\alpha^2\beta)_k$	C_6	con.
38b*	$6m^2(r^2-r+1)$	$\alpha^3=\beta^2=(\alpha\beta)^6=1$	$(\alpha^2\beta\alpha\beta)_k$	$(\alpha\beta\alpha^2\beta)_k Y^m X^{mr}=1$ $k=m(r^2-r+1)$	C_6	con.
39a	$6k^2$	$\alpha^3=\beta^2=(\alpha\beta\alpha^2\beta)^3=1$	$(\alpha^2\beta\alpha^2\beta)_k$	$(\alpha\beta\alpha\beta)_k$	G_6	alt.
39b	$6m^2(r^2-r+1)$	$\alpha^3=\beta^2=(\alpha\beta\alpha^2\beta)^3=1$	$(\alpha^2\beta\alpha^2\beta)_k$	$(\alpha\beta\alpha\beta)_k Y^m X^{mr}=1$ $k=m(r^2-r+1)$	G_6	alt.

Net (6.3.6.3).

No.	Order.	Im kleinen relations.	X	Y	Arrows.	
40a*	$3k^2$	$\alpha^3=\beta^3=(\alpha\beta)^3=1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k$	C_3	con. con.
40b*	$3m^2(r^2+r+1)$	$\alpha^3=\beta^3=(\alpha\beta)^3=1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k Y^m X^{mr}=1$ $k=m(r^2+r+1)$	C_3	con. con.
41	24	$\alpha^3=\beta^3=\alpha\beta\alpha^2\beta\alpha\beta^2=1$				alt. alt.

Net (6.4.3.4).

No.	Order.	Im kleinen relations.	X	Y	Arrows.	
42a*	$6k^2$	$\alpha^6=\beta^3=(\alpha\beta)^2=1$	$(\alpha^2\beta^2)_k$	$(\alpha^4\beta)_k$	C_6	con. con.
42b*	$6m^2(r^2-r+1)$	$\alpha^6=\beta^3=(\alpha\beta)^2=1$	$(\alpha^2\beta^2)_k$	$(\alpha^4\beta)_k Y^m X^{mr}=1$ $k=m(r^2-r+1)$	C_6	con. con.

Net (12.6.4).

No.	Order.	Im kleinen relations.	X	Y	Arrows.	
43a	$12k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\gamma)^6=(\alpha\beta)^3=(\beta\gamma)^2=1$	$[(\alpha\beta\gamma)^2]_k$	$[(\beta\alpha\gamma)^2]_k$	$G_{12}{}^5$	
43b	$36k^2$	$\alpha^2=\beta^2=\gamma^2=(\alpha\gamma)^6=(\alpha\beta)^3=(\beta\gamma)^2=1$	$[(\alpha\beta\gamma)^2]_{3k}$	$[(\beta\alpha\gamma)^2]_{3k} Y^k X^k=1$	$G_{12}{}^5$	

Groups of genus one.

No.	Order.	Im kleinen relations.	X	Y		Arrows.
Ia	$2kl$	$\alpha^2=\beta^2=\gamma^2=\delta^2=\alpha\beta\gamma\delta=(\alpha\beta\gamma)^2=1$	$(\alpha\delta)_k$	$(\alpha\beta)_l$	C_2	
Ib	$2mtr$	$\alpha^2=\beta^2=\gamma^2=\delta^2=\alpha\beta\gamma\delta=(\alpha\beta\gamma)^2=1$	$(\alpha\delta)_{mr}$	$(\alpha\beta)_{mt}Y^tX^r=1$	C_2	
IIa	$3k^2$	$\alpha^3=\beta^3=\gamma^2=\alpha\beta\gamma=(\alpha\beta)^2=1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_k$	C_3	con. con.
IIb	$3m^2(r^2-r+1)$	$\alpha^3=\beta^3=\gamma^2=\alpha\beta\gamma=(\alpha\beta)^2=1$	$(\alpha\beta^2)_k$	$(\beta^2\alpha)_kY^mX^{mr}=1$ $k=m(r^2-r+1)$	C_3	con. con.
IIIa	$4k^2$	$\alpha^2=\beta^4=\gamma^4=\alpha\beta\gamma=(\alpha\beta)^4=1$	$(\beta^3\gamma)_k$	$(\beta\gamma^3)_k$	C_4	con. con.
IIIb	$4m^2(r^2+1)$	$\alpha^2=\beta^4=\gamma^4=\alpha\beta\gamma=(\alpha\beta)^4=1$	$(\beta^3\gamma)_k$	$(\beta\gamma^3)_kY^mX^{mr}=1$ $k=m(r^2+1)$	C_4	con. con.
IVa	$6k^2$	$\alpha^2=\beta^3=\gamma^6=\alpha\beta\gamma=(\beta\gamma)^2=1$	$(\gamma^2\beta^2)_k$	$(\gamma^4\beta)_k$	C_6	con. con.
IVb	$6m^2(r^2+r+1)$	$\alpha^2=\beta^3=\gamma^6=\alpha\beta\gamma=(\beta\gamma)^2=1$	$(\gamma^2\beta^2)_k$	$(\gamma^4\beta)_kY^mX^{mr}=1$ $k=m(r^2+r+1)$	C_6	con. con.

Numbers 15, 16, 29, and 31 also occur with a non-oriented figure.

11. There are of course many isomorphisms of the groups though not of the representations. The starred numbers have been mentioned. The possession of the same order, the same self-conjugate abelian subgroup and the same quotient group is not sufficient for isomorphism. For example (2b) and (11b) agree in these respects but if all the integers are equal to two the groups of order 32 are distinct, (11b) has operators of order 8 while (2b) has not.

12. I add a list of groups of low order and the classes in which they occur.

Groups of order 8.

(2)	$C_2.C_4$	6, 10, 11, 12, 14, 15, 19, 26, 27, 31, 32, 34, 36.
(3)	$C_2.C_2.C_2$	2, 3, 21, 22, 34.
(4)	$G_8{}^4$	11, 22, 29, 33, 37.
(5)	Q_8	13, 16, 17.

Groups of order 10.

(1)	$C_2.C_5$	9.
(2)	$G_{10}{}^5$	10.

Groups of order 12.

(1)	C_{12}	9.
(2)	$C_2.C_6$	9.
(3)	$G_{12}{}^7 3$	10, 32.
(4)	$G_{12}{}^4$	40.
(5)	$G_{12}{}^5$	2, 3, 21, 27, 43.

Groups of order 14.

(1)	C_{14}	9.
(2)	$G_{14}{}^7$	10.

Groups of order 16.

(2)	$C_8.C_2$	9, 15, 36.
(3)	$C_4.C_4$	7, 9.
(4)	$C_4.C_2.C_2$	2, 5′.
(5)	$C_2.C_2.C_2.C_2$	1.
(6)	Burnside III vi	10, 12, 15, 20, 30.
(7)	" III vii	2, 4.
(8)	" III viii	8, 10.

(9)	Burnside III ix	2, 3, 5, 21, 22, 23, 34, 35.
(10)	" III x	6, 11, 26, 32, 33.
(11)	" VI	absent.
(12)	" VII	11, 21, 37.
(13)	" VIII	11, 27, 28.
(14)	" IX	11.

Groups of order 18.

(1)	C_{18}	9.
(2)	$C_6.C_3$	9, 12, 42.
(3)	$G_{18}{}^6 1$	2, 3, 10, 25, 26, 38, 39.
(4)	$G_{18}{}^6 2$	21, 24.
(5)	$\alpha^9 = \beta^2 = 1 \quad \alpha\beta = \beta\alpha^8$	absent.

Groups of order 20.

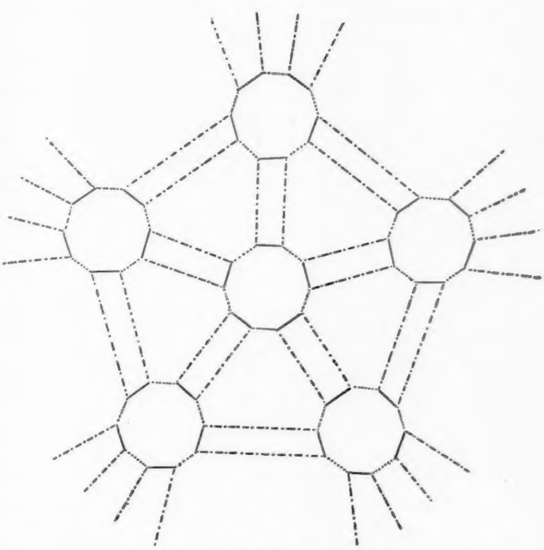
(1)	C_{20}	9.
(2)	$C_{10}.C_2$	2, 9, 10, 15, 26.
(3)	$G_{20}{}^5$	6, 32.
(4)	$G_{20}{}^7$	2, 11, 21, 26, 27.
(5)	$\alpha^5 = \beta^4 = 1 \quad \alpha\beta = \beta\alpha^4$	10.

Groups of order 24.

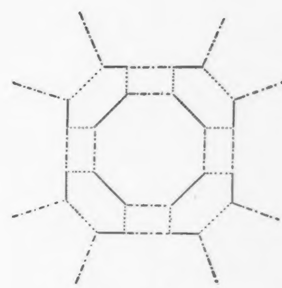
(1)	C_{24}	9.
(2)	$C_{12}.C_2$	9, 10, 15, 22, 36.
(3)	$C_3.C_2.C_2.C_2$	2.
(4)	$G_8{}^4.C_3$	10, 26.
(5)	$Q_8.C_3$	2, 10, 26.
(6)	Burnside I	10.
(7)	" II_1	2, 26.
(8)	" II_2	10.
(9)	" III_1	25, 42.
(10)	" III_2	1, 2, 3, 9, 21, 22.
(11)	" IV_1	41.
(12)	" IV_2	absent.
(13)	" V_1	3, 11, 21, 26, 27, 37.
(14)	" V_2	11, 22, 27.
(15)	" $\mathrm{V}_3 = G_{24}{}^4$	24, 39.

The plates give a selection of the drawings of the groups in question. Plate I has the three groups of the projective plane. Plates II, III, IV, V contain groups on the anchor ring and at least one example for every net.

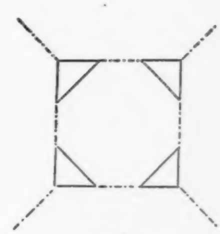
PLATE I.



G_{60}^{5}

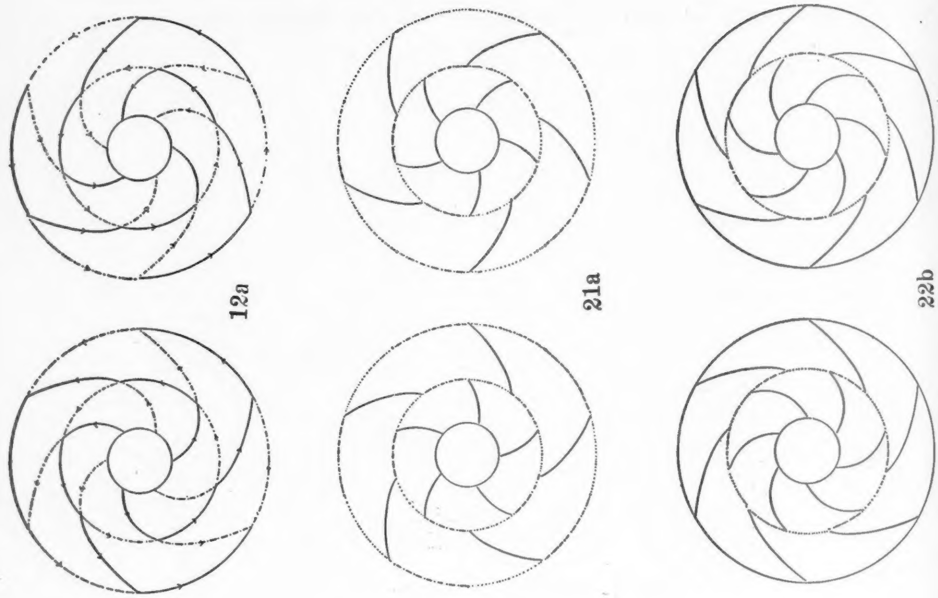


G_{24}^{4}



G_{12}^{4}

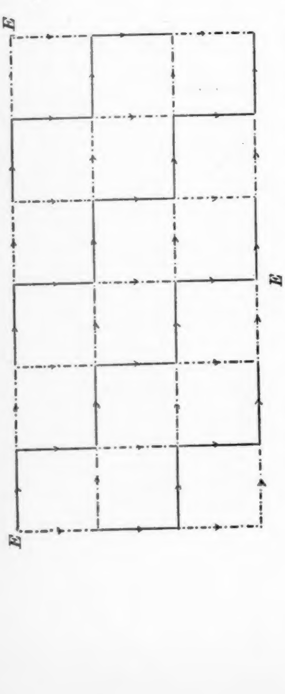
PLATE II.



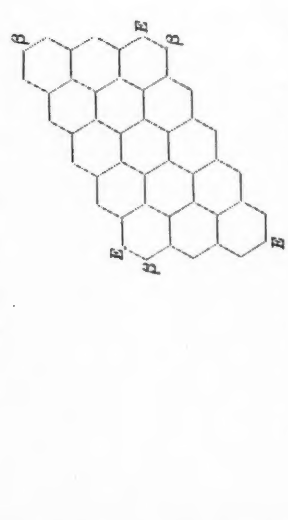
12a

21a

22b

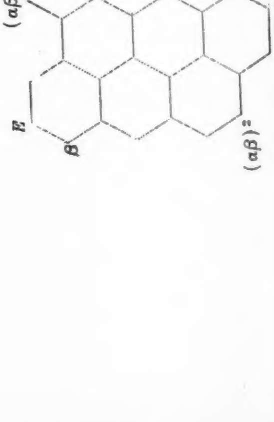


12a $2k^2$, $k = 3$ $\alpha^{2k} = \beta^{2k} = 1$ $\alpha^2 = \beta^2$



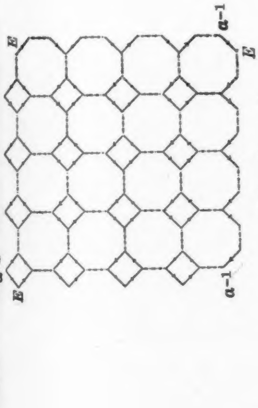
2kl $k = 5$ $l = 3$ $\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta\gamma)^2 = E$

21a $(\alpha\beta)_k$, $(\beta\gamma)_l$, G_{30}



8kl. $k = 1$, $l = 3$ $\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma\alpha\gamma\beta = E$

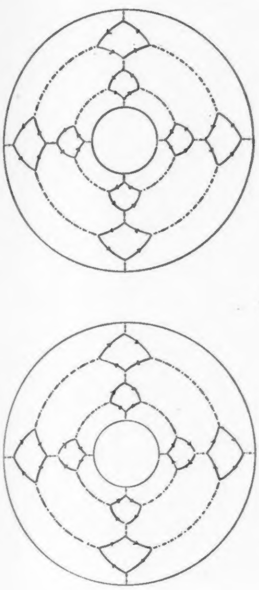
PLATE III.



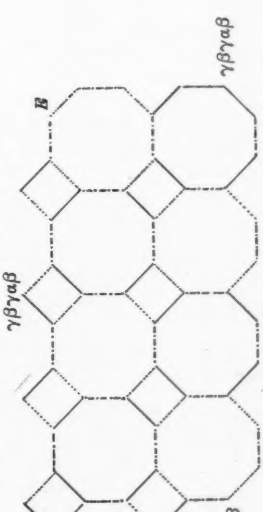
32a

$4k^2 \quad k = 4$

$\alpha^4 = \beta^2 = (\alpha\beta)^4 = E \quad (\alpha^2\beta)_k, \ (\alpha\beta\alpha)_k \quad G\,64$



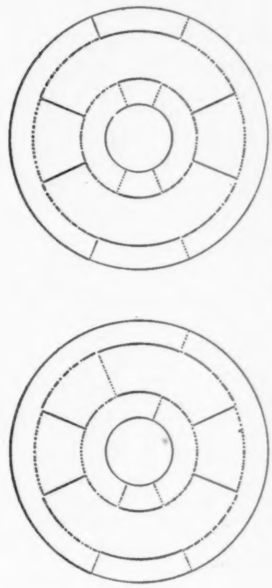
32a



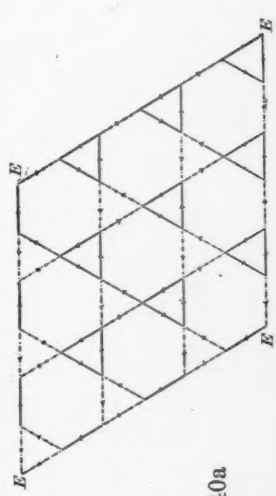
$8k^2, \quad k = 2$

$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^2 = (\alpha\beta)^4 = (\beta\gamma)^4 = E$

35b $\quad (\alpha\beta\gamma)_{2k}, \ (\gamma\beta\alpha)_{2k}, \ (\gamma\beta\alpha)^k = (\alpha\gamma\beta)^k \quad G\,32$

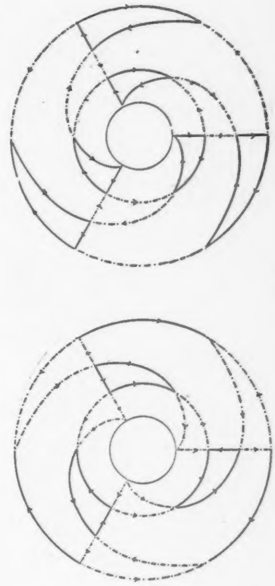


35b



40a

$3k^2 \ k = 3 \quad \alpha^3 = \beta^3 = (\alpha\beta)^3 = E \quad (\alpha\beta^2)_k, \ (\beta^2\alpha)_k \quad G\,27$

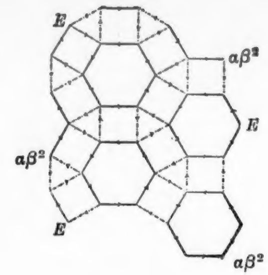


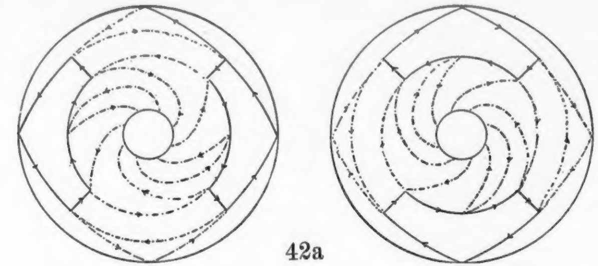
40a

PLATE IV.

$$6k^2 \quad k = 2$$
$$\alpha^6 = \beta^3 = (\alpha\beta)^2 = E$$
$$(\alpha^2\beta^2)_k, \; (\alpha^4\beta)_k$$
$$G\ 24.6.1$$

42a

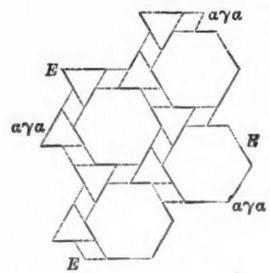


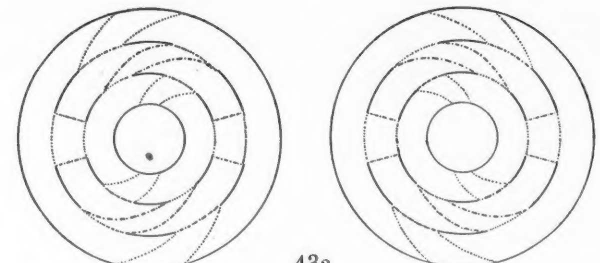


42a

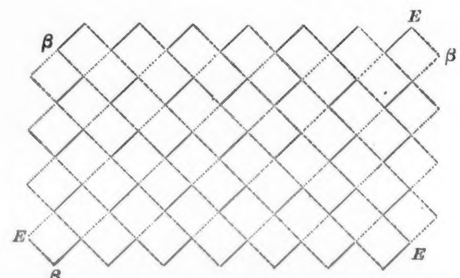
$$12k^2 \quad k = 2$$
$$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\gamma)^6 = (\alpha\beta)^3 = (\beta\gamma)^2$$
$$[(\alpha\beta\gamma)^2]_k, \; [(\beta\alpha\gamma)^2]_k$$
$$G\ 48$$

43a



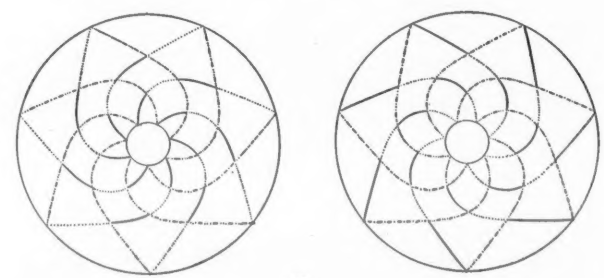


43a



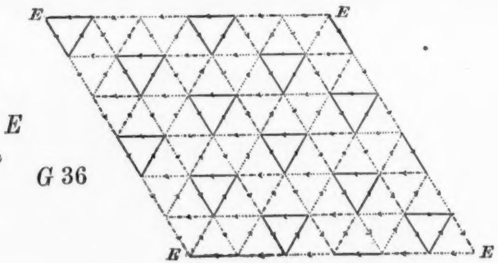
$$p = 1 \quad 2mn \quad m = 7 \quad n = 4$$
$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \alpha\beta\gamma\delta = E$$
$$(\alpha\beta)_m, \; (\beta\gamma)_n \qquad G\ 56$$

Ia



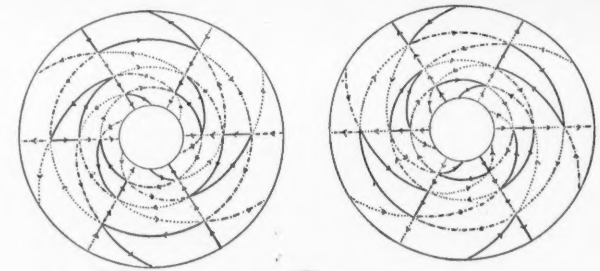
Ia

Plate V.

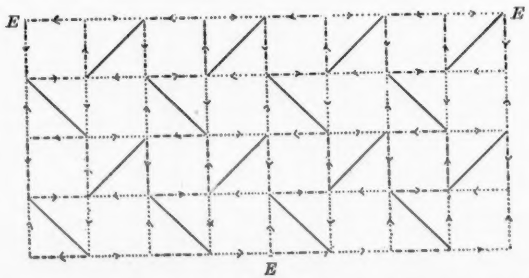


$p=1 \quad 9m^2 \quad m=2$
$\alpha^3=\beta^3=\gamma^3=\alpha\beta\gamma=E$
$(\alpha\beta^2)_{3m},\ (\beta^2\alpha)_{3m}$
$(\alpha\beta^2)^m=(\beta^2\alpha)^m \qquad G\ 36$

IIb

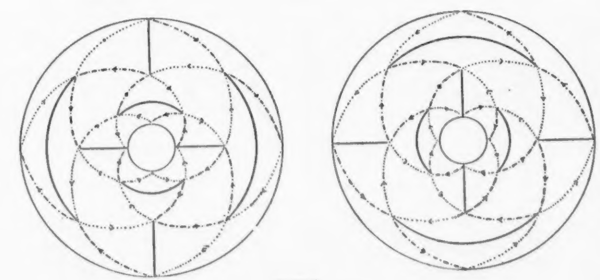


IIb

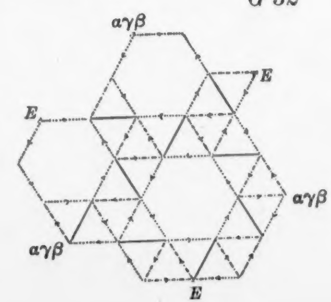


$p=1 \quad 8m^2 \quad m=2 \quad \alpha^2=\beta^4=\gamma^4=\alpha\beta\gamma=E$
$(\beta^3\gamma)_{2m},\ (\beta\gamma^3)_{2m} \quad (\beta^3\gamma)^m.(\beta\gamma^3)^m=E$
$G\ 32$

IIIb

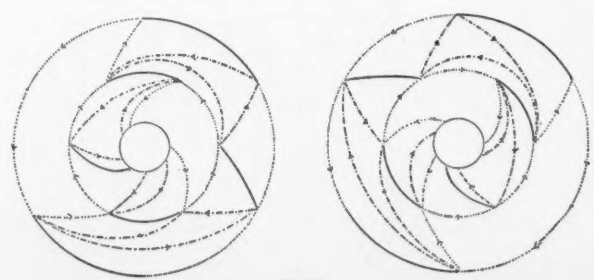


IIIb



$18m^2 \quad m=1$
$\alpha^2=\beta^3=\gamma^6=\alpha\beta\gamma=E$
$(\gamma^2\beta^2)_{3m},\ (\gamma^4\beta)_{3m}$
$(\gamma^2\beta^2)^m.(\gamma^4\beta)^m=E$
$G\ 18$

IVb



IVb

CONCERNING CONTINUOUS IMAGES OF THE INTERVAL.

By G. T. Whyburn.

1. This paper is concerned with the following celebrated theorem of Hahn-Mazurkiewics: *

(A). *Every compact, metric and locally connected continuum is the image under a continuous transformation of the unit interval.*

At the Bologna congress † in 1928, Hahn gave a new proof for this theorem which (to quote him exactly), "nicht nur die ursprünglichen Beweise von Herrn Mazurkiewicz und mir, sondern auch die zeither von verscheidenen Seiten mitgeteilten Beweise an Einfachheit und Durchsichtigkeit übertrifft, indem er die Behauptung als unmittelbar Folge bekannter, mit elementaren Mitteln beweisbar Sätze aufweist. Es sind dies die folgenden Sätze:" . . . (There follows the statements of three theorems, of which the second is here quoted.)

II. *Every two points a and b of a self-compact, connected and locally connected set M can be joined by a subset M′ of M which is the continuous image of an interval.*

With the aid of the three theorems quoted, Hahn proceeds to prove (A) with very little difficulty.

The author has found that by a slight modification of Hahn's proof ‡ for II alone, one can obtain a complete proof for (A). This modification adds little if any to the complexity and very little to the length of the proof for II; and it will be noted that, aside from the separability of the space and the lemma given below in § 2, use is made of no concept or property which was not used by Hahn in proving II. We therfore obtain a proof for (A) which is in every way as simple and elementary as the proof required for II alone and which *completely avoids* the other two theorems used by Hahn in his simplified proof. Thus our proof, which will be given in detail in § 3 below, seems to be

* See Hahn, *Wiener Berichte*, Vol. 123 (1914), p. 2433; Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 1 (1920), p. 166, and note reference there given to an earlier paper by Mazurkiewicz.

† See the proceedings of this congress, Vol. 2, p. 217.

‡ See Hahn, *Wiener Berichte*, *loc. cit.*, p. 2436.

more direct and elementary than any which has yet been given for the very fundamental theorem (A).

2. *Definition.* If ϵ is any positive number, then by an ϵ-chain of points joining two given points a and b is meant a finite sequence of points $a = X_0$, $X_1, X_2, \ldots, X_n = b$ such that the distance between any two successive points in this sequence is $< \epsilon$.

It is well known that if a set is connected, it contains, for every $\epsilon > 0$, an ϵ-chain of points joining any two of its points. We shall make use of the following immediate consequence of this fact.

LEMMA. *If F is any finite subset of a connected set M and a and b are any two points of F, then for every $\epsilon > 0$, M contains an ϵ-chain of points joining a and b and containing all points of F.*

For let the points of F be $a = p_1, p_2, \ldots p_n = b$. Then to obtain an ϵ-chain of points of M from a to b we first take such a chain from $a = p_1$ to p_2, then such a chain from p_2 to p_3, then one from p_3 to p_4, and so on until we reach $p_n = b$. Clearly the sequence of points obtained in this order is an ϵ-chain of points from a to b containing all points of F.

3. *The Proof.* Let $P = \sum_1^\infty p_i$ be a countable set of points which is dense in M and, for each integer n, let $P_n = \sum_1^n p_i$. Let $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ be a sequence of positive numbers such that $\sum_1^\infty \epsilon_i$ converges. Since M is uniformly locally connected,* there exists, for each of these numbers ϵ_k, a positive δ_k such that every two points x and y of M whose distance apart is $< \delta_k$ lie together in a connected subset of M of diameter $< \epsilon_k/2$.

Now let a and b be any two points of M. There exists an integer n_1 such that every point of M is at a distance $< \delta_1$ from some point of P_{n_1}. Since M is connected, it follows by the lemma that there exists in M a δ_1-chain C^1 of points joining a and b and containing P_{n_1} and which we may suppose consists of exactly $2^{v_1} + 1$ points

$$a = X_0^1, X_1^1, \cdots X^1_{2^{v_1}} = b. \tag{1}$$

* See Hahn, *loc. cit.*, p. 2435. This property is proved as follows: If on the contrary, for some $\epsilon > 0$, there exists no such δ, it follows that there exist two infinite sequences of points $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ each converging to the same point p of M and such that for no n can x_n and y_n be joined by any connected subset of M of diameter $< \epsilon$. But clearly this contradicts the fact that M is locally connected at the point p.

There exists an integer $n_2 \geqq n_1$ such that every point of M is at a distance $< \delta_2$ from some point of P_{n_2}. For each integer i, $0 \leqq i \leqq 2^{v_1}$, let F_i^1 be the set of all those points of P_{n_2} whose distances from X_i^1 are $< \delta_1$. Since the chain (1) contains P_{n_1}, it follows that $P_{n_2} = \sum_0^{2^{v_1}} F_i^1$. Now for each i, M contains a connected set M_i^1 of diameter $< \epsilon_1$ which contains $X_i^1 + F_i^1 + X^1{}_{i+1}$, because each point of $F_i^1 + X^1{}_{i+1}$ is at a distance $< \delta_1$ from the point X_i^1. Hence by the lemma, M_i^1 contains a δ_2-chain of points C_i^1 joining X_i^1 and $X^1{}_{i+1}$ and containing all points of F_i^1. We may suppose that all of these chains C_i^1 contain the same number, say $2^{v_2} + 1$, of points which we shall denote by

$$X_i^1 = X^2{}_{i \cdot 2^{v_2}}, X^2{}_{i \cdot 2^{v_2}+1}, \cdots, X^2{}_{(i+1)2^{v_2}} = X^1{}_{i+1}.$$

Clearly the chains $[C_i^1]$ taken in the order $C_1^1, C_2^1, \cdots, C^1{}_{2^{v_1}}$ form a δ_2-chain C^2 from a to b which contains all points for P_{n_2}.

Let us continue in this way. In general, for each k, we choose an integer $n_k \geqq n_{k-1}$ such that every point of M is at a distance $< \delta_k$ from some point of P_{n_k}. For each i, $0 \leqq i \leqq 2^{v_1+v_2+\cdots+v_{k-1}}$, let F_i^{k-1} be the set of all points of P_{n_k} whose distances from the point X_i^{k-1} are $< \delta_{k-1}$. Then M contains a connected set M_i^{k-1} of diameter $< \epsilon_{k-1}$ which contains $X_i^{k-1} + F_i^{k-1} + X^{k-1}_{i+1}$; and by the lemma, M_i^{k-1} contains a δ_k-chain of points C_i^{k-1} joining X_i^{k-1} and X^{k-1}_{i+1} and containing all points of F_i^{k-1}. We may suppose that all of these chains $[C_i^{k-1}]$ contain the same number, say $2^{v_k} + 1$, of points. Clearly then, these chains taken in the order $C_1^{k-1}, C_2^{k-1}, \cdots C_{2^{v_1+v_2+\cdots+v_{k-1}}}$ form a δ_k-chain C^k of points from a to b which contains P_{n_k} and consists of exactly $2^{v_1+v_2+\cdots+v_k} + 1$ points:

$$a = X_0^k, X_1^k, \cdots, X^k{}_{2^{v_1+v_2+\cdots+v_k}} = b,$$

in which notation we have always:

$$X^k{}_{i \cdot 2^{v_k}} = X_i^{k-1}. \tag{2}$$

Now to the values

$$t = i/2^{v_1+v_2+\cdots+v_k} \qquad (i = 0, 1, \cdots, 2^{v_1+v_2+\cdots+v_k})$$

of the parameter t we make correspond the points X_i^k, respectively, and set $T(t) = X_i^k$, so that we thus define our transformation T for the set D all values of $t (0 \leqq t \leqq 1)$ which are dyadically representable, i. e., which can be written as factions having powers of 2 for denominators. It is a consequence of (2) that T is single valued on D.

We shall now show that the transformation T thus defined on D is uniformly continuous on D. Let ϵ be any positive number and let us choose k so large that

$$(3) \qquad \sum_{k}^{\infty} \epsilon_i < \epsilon/2$$

and set

$$(4) \qquad \delta = 1/2^v, \text{ where } v = v_1 + v_2 + \cdots + v_k.$$

Then if t_1 and t_2 are any two values of t in D such that $| t_1 - t_2 | < \delta$, they must lie between three successive values,

$$(j-1)/2^v, \quad j/2^v, \quad (j+1)/2^v$$

of t in D. Let us suppose t_1 lies between the last two of these values. We can write $t_1 = m/2^{v+u}$, where $u = v_{k+1} + \cdots + v_{k+w}$. Thus we have $T(t_1) = X_m^{k+w}$. Now the point X_m^{k+w} was arrived at in the following manner. We defined the chain C_j^k of points joining the two points X_j^k and $X^k{}_{j+1}$ and lying wholly in the ϵ_k-neighborhood of the point X_j^k; then joining some two points of C_j^k there was set up the chain $C_{j_1}^{k+1}$ so that it lay wholly in the ϵ_{k+1}-neighborhood of each of these points and hence lay in the $\epsilon_k + \epsilon_{k+1}$-neighborhood of X_j^k. Between some two points of $C_{j_1}^{k+1}$ there was set up the chain of points $C_{j_2}^{k+2}$ lying wholly in the ϵ_{k+2}-neighborhood of each of these points and hence in the $\epsilon_k + \epsilon_{k+1} + \epsilon_{k+2}$-neighborhood of X_j^k, and so on. After w steps of this sort we reach the point X_m^{k+w}, which therefore lies in the $\epsilon_k + \epsilon_{k+1} + \cdots + \epsilon_{k+w}$-neighborhood of X_j^k. By virtue of (3), we have

$$\rho[T(t_1), X_j^k] = \rho[X_m^{k+w}, X_j^k] < \epsilon/2.*$$

In exactly the same manner we prove that

$$\rho[T(t_2), X_j^k] < \epsilon/2.$$

These two relations give at once that $\rho[T(t_1), T(t_2)] < \epsilon$, which proves that T is uniformly continuous on D.

Now by a well known theorem, it is possible † to extend the definition of

---

* We employ the usual notation $\rho(x, y)$ for the distance between the points x and y.

† This is done as follows. Let x be any limit point of D and let $x_1, x_2, x_3, \cdots$ be any sequence of points of D converging to x. By virtue of the uniform continuity of T on D it follows that the image points $x'_1, x'_2, x_3', \cdots$ under T of the points $x_1, x_2, x_3, \cdots$, respectively, converge to some point x' of M. We then set $T(x) = x'$. Now if $y_1, y_2, y_3, \cdots$ is any sequence of points of D converging to x, it follows that since $\rho(x_n, y_n) \to 0$ with $1/n$, $\rho(x'_n, y'_n)$ also $\to 0$ with $1/n$ and hence that the sequence $y'_1, y'_2, y'_3, \cdots$ also converges to the point x'. Thus the extended transformation is single valued; and since we get the same conclusion if $y_1, y_2, y_3, \cdots$ is any sequence of points of $\overline{D}$, it follows that T is also continuous.

the transformation T to the limit points of D and thus to the entire interval $(0, 1) = I$ in such a way that the extended transformation T is single valued and continuous and has the same values on D as before. Now since $T(D) = \sum_1^\infty C^n$, and for each n, $C^n \supset P_{n_n} = \sum_1^{n_n} p_i$, it follows that $T(D) \supset \sum_1^\infty p_i = P$; and since P is dense in M, it follows that $T(I) = M$. Thus M is the image under the extended transformation T of I, and the proof is complete.

4. *Conclusion.* The author wishes to emphasize the small amount he has had to contribute in order to obtain the proof of (A) from the proof of II. Indeed, the essential change consists in altering the choice of a certain sequence of points in the proof of II in such a way as to insure that the subset M' of M which joins a and b and is the continuous image of the interval shall be identical with M. Thus our proof is really only a translation of Hahn's proof for II slightly modified in places so as to attain this more advantageous state of affairs and thus establish (A).

THE JOHNS HOPKINS UNIVERSITY.

ON QUASI-METRIC SPACES.*

By W. A. Wilson.

1. Let Z be a class of elements such that for each pair x and y there are two non-negative numbers called the distances from x to y and from y to x. These are designated by xy and yx and satisfy these axioms:

I. $xy = 0$ if and only if $x = y$.

II. $xz \leqq xy + yz$.

The same relations hold for yx, zy, and zx. Such a space will be called quasi-metric.†

If $xy = yx$, the common value is denoted by $\overline{xy}$; if this is true for every pair, Z is metric. Thus in one sense a quasi-metric space is merely the result of suppressing the axiom that $xy = yx$ from the definition of metric space. Usually the result of such a limitation on a set of axioms is to diminish the number of theorems easily deducible, but in this case there is an embarrassing richness of material. The first half of this paper (§§ 1-5) contains some of the properties of quasi-metric spaces, and relations between quasi-metric, metric, and topological spaces are discussed in the latter half (§§ 6-8).

As an example of a quasi-metric space consider any metric space M decomposed into mutually disjoint bounded closed sets. Let Z be the aggregate whose elements are these closed sets and for any pair let xy be the lower bound of the numbers $\{r\}$ such that every point of y has a (metric) distance from the set x less than r. Then Z is quasi-metric.

2. *Limiting points.* Let $A = \{x\}$ be a set in Z and a a point such that for every $r > 0$ there is at least one point x of A distinct from a which satisfies the relation $ax < r$. Then a is called a *u-limiting point* of A. If for at least one point x of $A - a$, $xa < r$, a is called an *l-limiting point* of A. If for at least one point x of $A - a$, both $ax < r$ and $xa < r$, a is called a *c-limiting point* of A. It is sometimes convenient to call u-limiting and l-limiting points collectively *quasi-limiting* points. In the example of § 1 let $A = \{x\}$ be a set of elements of Z and a some other element of Z; also

* Presented to the Society, September 9, 1930.

† Asymmetric definitions of distance have been used by various authors before (e.g., F. Hausdorff, *Mengenlehre*, pp. 145-146), but, as far as the writer knows, the properties of quasi-metric spaces have never been completely worked out.

let M be compact. If in M the set a contains the upper closed limiting set (in the Hausdorff sense) of a sequence $\{x_i\}$ chosen from A, then in Z the *point* a is a u-limiting point of A. If a is contained in the lower closed limiting set of $\{x_i\}$, a is an l-limiting point of A in Z. If a is the closed limiting set of $\{x_i\}$, a is a c-limiting point of A in Z.

It follows at once that for a to be a u-limiting, l-limiting, or c-limiting point of $A = \{x\}$, it is necessary and sufficient that for each $r > 0$, there is an infinity of points $\{x\}$ of A such that $ax < r$, $xa < r$, or $ax < r$ and $xa < r$, respectively.

A sequence $\{x_i\}$ is said to have the point a as a *u-limit, l-limit,* or *c-limit,* if for each $r > 0$ there is an i' such that for every $i > i'$, $ax_i < r$, $x_ia < r$, or $ax_i < r$ and $x_ia < r$, respectively. If from some i on the points $\{x_i\}$ are distinct, any limit of the sequence $\{x_i\}$ is of course the corresponding kind of a limiting point of the set $\{x_i\}$.

Simple examples show that a sequence may have any number of quasi-limits of either kind and that the Cauchy convergence criterion is not necessary for the existence of semi-limits. To see this, let $Z = A + B + C$, where A is the set of positive real numbers, B the set of negative real numbers, and C is any set whatever containing no real numbers. If $x + y \subset A$ or $x + y \subset B$, let $xy = yx = |x - y|$; if $x + y \subset C$, let $xy = yx = 1$; if x lies in A and y in B, let $xy = 1 + |x - y|$ and $yx = |x - y|$; if x lies in C and y in A, let $xy = |y|$ and $yx = 1 + |y|$; if x lies in C and y in B, let $xy = 1 + |y|$ and $yx = |y|$. Then Z is quasi-metric and, if $\{x_i\}$ is a sequence chosen from $A + B$ such that $|x_i| \to 0$, every point of C is a u-limit of $\{x_i\}$ when $\{x_i\} \subset A$ and an l-limit when $\{x_i\} \subset B$. Upper semi-continuous decompositions of compact continua usually furnish examples of the second phenomenon when the distances are defined as in § 1. However, we have the following theorems.

THEOREM I. *If a point a is both a u-limit and an l-limit of the sequence $\{x_i\}$, it is the only limit of any kind and the Cauchy criterion is satisfied.*

Proof. By hypothesis, for any $r > 0$, there is an i' such that $ax_i < r/2$ and $x_ia < r/2$ for every $i > i'$. If also $j > i'$, $ax_j < r/2$; hence $x_ix_j \leqq x_ia + ax_j < r$ for i and j both greater than i' and the Cauchy criterion is satisfied. If b is any u-limit of the sequence, there is an i'' such that $bx_i < r/2$ for $i > i''$. Taking i greater than both i' and i'', we have $ba \leqq bx_i + x_ia < r$, whence $b = a$. Thus the sequence has only one u-limit and in like manner a is the only l-limit.

COROLLARY. *No sequence has more than one c-limit.*

Note that in general a point a may be both a u-limiting and an l-limiting point of a set A, although not a c-limiting point.

THEOREM II. *If a is a u-limit or an l-limit of the sequence $\{x_i\}$, but is not the c-limit of the sequence, then $\{x_i\}$ has no l-limit or u-limit, respectively.*

Proof. Let a be a u-limit of $\{x_i\}$. If there were an l-limit b, we would have for any $r > 0$ an i', such that $ax_i < r/2$ and $x_ib < r/2$ for every $i > i'$. But then $ab < r$, or $b = a$.

COROLLARY. *If a sequence $\{x_i\}$ has more than one u-limit or l-limit, it has no l-limit or u-limit, respectively.*

THEOREM III. *Let a and c be fixed points and let x be a variable point. If $cx \to 0$, $\limsup ax \leqq ac$ and $\liminf xa \geqq ca$. If $xc \to 0$, $\liminf ax \geqq ac$ and $\limsup xa \leqq ca$. If both $cx \to 0$ and $xc \to 0$, then $ax \to ac$ and $xa \to ca$.*

The proofs of these statements come directly from Axiom II. The theorem shows that in general the distance functions are not continuous at semi-limiting points, but are semi-continuous. Both, however, are continuous at c-limiting points.

3. *Closed sets and regions.* A set is called *u-closed, l-closed,* or *c-closed* if it contains all of its u-limiting, l-limiting, or c-limiting points, respectively.

A *u-sphere, l-sphere,* or *c-sphere* of center a and radius r is the set of points $\{x\}$ such that $ax < r$, $xa < r$, or $ax < r$ and $xa < r$, respectively. These are denoted by $U_r(a)$, $L_r(a)$, and $S_r(a)$, respectively. Note that $S_r(a) = U_r(a) \cdot L_r(a)$.

A set A such that for each of its points $\{a\}$ some $U_r(a)$, $L_r(a)$, or $S_r(a)$ is wholly contained in A is called a *u-region, l-region,* or *c-region,* respectively.

For each type of closed set and region we have at once the usual theorems regarding divisors, unions, and complements. In particular, if A', A'', or $\bar{A}$ denotes the union of the set A and its u-limiting, l-limiting, or c-limiting points, respectively, A' is u-closed, A'' is l-closed, and $\bar{A}$ is c-closed. In addition there are various other properties due to the relations between the three types of limiting points.

THEOREM I. *Every quasi-closed set or quasi-region is a c-closed set or c-region, respectively, but not conversely.*

Proof. To fix the ideas let A be u-closed. If a is a c-limiting point of A, it is a u-limiting point by the definitions and so belongs to A. Hence A is

c-closed. The converse is not true, because a u-limiting point is not necessarily a c-limiting point. The statement regarding regions follows from the complementary relations between regions and closed sets.

Consequently the quasi-closed sets are special types of c-closed sets. Constant vigilance is necessary to avoid confusing this with the fact that c-limiting points are special types of quasi-limiting points.

The u-spheres and l-spheres have certain features not analogous to those of metric spheres. Although a u-sphere or l-sphere of center a and radius r is a u-region or l-region, respectively, the set of points for which $ax > r$ or $xa > r$, respectively, is an l-region or u-region, respectively. The set for which $ax = r$ or $xa = r$ is the divisor of a u-closed and an l-closed set, and so merely c-closed. Finally, if $U = U_r(a)$ and $L = L_r(a)$, U' may contain points for which $ax > r$ and L'' points for which $xa > r$; in fact, a point b may be a u-limiting point of $U_r(a)$ for every r. Among the various relations between semi-closed sets and semi-regions suggested by these facts, the following generalization of the well-known separation theorem for metric spaces may be of interest.

THEOREM II. *Let A and B lie in a semi-metric space and $A' \cdot B + A \cdot B'' = 0$. Then there is an l-region R and a u-region S such that $A \subseteq R$, $B \subseteq S$, $R' \cdot S + R \cdot S'' = 0$, and $A' \cdot B'' = R' \cdot S''$.*

Proof. Since $A \cdot B'' = 0$, there is for each point x of A an $r > 0$ and less than one-third the lower bound of yx as y ranges over B''. Enclose x in an l-sphere L_x of center x and radius r. If R denotes the union of these spheres, it is an l-region and contains A.

Now $B'' \cdot R = 0$ by construction and, for each x, $B'' \cdot L_x' = 0$. For, if y lies in B'', $yx > 3r$, while if y lies in L_x', $yx \leqq r$. Hence if B contains a point y of R', y is a u-limit of a sequence $\{x_i\}$ of points, each in an l-sphere L_i of center a_i chosen from the set forming R and no two in the same L_i. Now $ya_i \leqq yx_i + x_ia_i$. If the radii $\{r_i\}$ have a lower bound, then $ya_i < 2r_i$ for i large enough, since $yx_i \to 0$. This is false, as $ya_i > 3r_i$. Hence for a partial sequence $r_i \to 0$ and so $ya_i \to 0$, which is another contradiction, as $A' \cdot B = 0$. Thus $R' \cdot B + R \cdot B'' = 0$.

Now enclose each point y of B in a u-sphere U_y of center y and radius r' less than one-third the lower bound of yx as x ranges over R'. If S is the union of these spheres, it is a u-region containing B and, as above, $R' \cdot S + R \cdot S'' = 0$.

Obviously $A' \cdot B'' \subseteq R' \cdot S''$. Let z lie in $R' \cdot S''$. Then z is not in R, but is a u-limit of a sequence $\{x_i\}$ of points of R, each lying in one of the l-spheres

forming R, say L_i with center a_i and radius r_i. Likewise z is an l-limit of a sequence of points $\{y_i\}$ of S, each lying in one of the u-spheres forming S, say U_i with center b_i and radius r_i'. As $zx_i \to 0$ and $y_i z \to 0$, $y_i x_i \to 0$. As in the earlier part of the proof there is a sub-sequence for which $r_i \to 0$ and $r_i' \to 0$. Then, since $za_i \leqq zx_i + x_i a_i \leqq zx_i + r_i$, $za_i \to 0$ and so z lies in A'. Likewise z lies in B''. Hence $R' \cdot S'' = A' \cdot B''$, which completes the proof.

COROLLARY. *If in the above A is u-closed, B is l-closed, and $A \cdot B = 0$, then also $R' \cdot S'' = 0$.*

4. The discussion of spheres in the previous section suggests the following sets of axioms:

Axiom III'. *For each pair of points a and b there is an $r > 0$ such that b does not lie in $U_r'(a)$.*

Axiom III''. *For each pair of points a and b there is an $r > 0$ such that b does not lie in $L_r''(a)$.*

Axiom IV'. *For each point a and each positive constant k there is an $r > 0$ such that, if $ab \geqq k$, b does not lie in $U_r'(a)$.*

Axiom IV''. *For each point a and each positive constant k there is an $r > 0$ such that, if $ba \geqq k$, b does not lie in $L_r''(a)$.*

Axiom III' is clearly equivalent to the statement that for each pair of points a and b there is an $r > 0$ such that there is no point x for which both $ax < r$ and $bx < r$, and hence to the statement that no sequence has more than one u-limit. Similar equivalences are of course valid for Axiom III''. Axioms IV' and IV'' are stronger forms of III' and III''.

5. If the quasi-metric space Z contains an enumerable set $E = \{c_i\}$ such that every point of Z is the c-limit of some sub-sequence chosen from E, we say that E is *dense* in Z and that Z is *separable.* We then have the usual theorems on the cardinal number of the set of regions and closed sets, including the Lindelöf covering theorem. Also, if a proper part M of Z is considered as a space, it is quasi-metric and separable.

In addition to these theorems it should be noted that in a separable quasi-metric space Z every point x which is not a c-limiting point of $Z - x$ must belong to the fundamental set E. Hence the set of points which are u-limiting, but not l-limiting points of Z, and vice versa, is enumerable.

On the other hand the imposition of separability on Z does not remove the possibility of a sequence having two or more u-limits or l-limits. As

an example, let Z be the sum of the plane sets C and A, where C is the set defined by $x=0$, $0<y\leqq 1/2$, and A the set defined by $0<x\leqq 1$, $y=0$, and let E be the set of points of Z which have both coördinates rational. If $a+b\subset C$ or $a+b\subset A$, let $ab=ba$ be the ordinary Cartesian distance. If a lies in C and b in A, let ab be the abscissa of b and $ba=ab+1$. Then Z is quasi-metric and every point is the c-limit of some sequence chosen from E, but every point of C is a u-limit of the sequence of points of A whose abscissae are $\{1/n\}$.

If Z is separable and also satisfies Axiom IV′, Theorem II of § 3 may be replaced by the following: *If $A'\cdot B+A\cdot B'=0$, there are disjoint u-regions R and S containing A and B respectively.* An analogous theorem corresponds to Axiom IV″. The method of proof is indicated in § 8.

6. Relations between quasi-metric and metric spaces. We first note that there is no way of defining distance so that there is a unique correspondence between the u-limiting (or l-limiting) points and points which are limiting points by the new distance definition. For we may have two distinct points, both of which are u-limits of the same sequence. But in a metric space no sequence has more than one limit.

If we define the distance between two points x and y as the quantity $\rho(x,y)=(xy+yx)/2$, we obtain a metric space Z' which has the same points as Z. For points where $\overline{xy}$ exists, $\rho(x,y)=\overline{xy}$; for other pairs of points $\rho(x,y)$ has a value between xy and yx. It is a simple matter to show that, if a is the limit in Z' of a sequence $\{x_i\}$, it is the c-limit of $\{x_i\}$ in Z; and conversely. A point which is not a c-limiting point of Z is an isolated point of Z' and, if Z is separable, the number of such points is enumerable.

7. Relations between quasi-metric and topological spaces. In Hausdorff's *Mengenlehre* (pp. 228-229) the topological axioms are listed in three groups: axioms of vicinities (A,B,C); separation axioms $(4,5,6,7,8)$; and cardinal number axioms $(9,10)$. This numbering will be used in the following theorems, which relate quasi-metric to topological spaces.

Theorem I. *Let Z be quasi-metric and let the vicinities of each point x be the u-spheres or l-spheres of center x and rational radii. Then Z is a topological space satisfying Axioms A, B, C, 4, and 9, but not necessarily satisfying Axiom 5. A u-limit, or l-limit, respectively, of a sequence is the topological limit of the sequence, and vice versa.*

Proof. The proof of the positive statements is immediate. That Z need not satisfy Axiom 5 follows from the fact that u-limits and l-limits may not be unique. In this and the following theorems analogous results are obtained

when the vicinities are c-spheres, but this case is not worth considering, as it was seen in § 6 that the space can be made metric with a preservation of c-limiting points.

THEOREM II. *Let Z be quasi-metric and separable, E being the enumerable set dense in Z. Let the vicinities be the u-spheres or l-spheres whose radii are rational and whose centers are points of E, and let the vicinity of a point x be any such sphere containing x. Then Z is a topological space satisfying Axioms A, B, C, 4, and 10, but not necessarily satisfying Axiom 5. A u-limit or l-limit, respectively, of a sequence is the topological limit of the sequence, and conversely.*

Proof. Consider the u-case; the other is similar. If a is any point of Z, there is a sub-sequence $\{c_n\}$ of E such that a is the c-limit of $\{c_n\}$; i. e., for any rational $r > 0$, $ac_n < r$ and $c_na < r$ for every n greater than some n'. Hence a lies in the u-sphere of center c_n and radius r. Thus Axiom A is satisfied; the validity of the other axioms readily follows.

Let a be any u-limit of a sequence $\{x_n\}$ and V be any vicinity of a. Now V is a u-sphere having some point c of E as its center and a rational radius r, and $ca < r$. Taking $r' < r - ca$, we have $ax_n < r'$ for every n larger than some n'. Hence $cx_n < ca + r' < r$, or V contains every x_n for $n > n'$.

Conversely, let a be the topological limit of the sequence $\{x_n\}$. Since E is dense in Z, there is for each rational $r > 0$ a point c of E such that $ac < r$ and $ca < r$. Then the u-sphere of center c and radius r is a vicinity of a and consequently contains every x_n for n larger than some n'. This gives $ax_n \leqq ac + cx_n < r + r = 2r$ for $n > n'$, which was to be proved.

THEOREM III. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 10. Then distances can be defined so that Z is quasi-metric and separable and, if a is any topological limit of the sequence $\{x_i\}$, then a is a u-limit of the sequence, and conversely.*

Proof. Let the enumerable set of vicinities be $\{V_i\}$. For any two points x and y and for each i, set $f_i(x, y) = 1$ if x lies in V_i and y in $Z - V_i$; otherwise $f_i(x, y) = 0$. Let $xy = \sum_1^\infty f_i(x, y)/2^i$. If $x = y$, every $f_i(x, y) = 0$ and $xy = 0$. If $x \neq y$, there is some V_i such that x lies in V_i and y in $Z - V_i$ by Axiom 4. Hence this $f_i(x, y) = 1$ and consequently $xy \neq 0$. Now let x, y, and z be any three points. If x is not in V_i, $f_i(x, z) = 0$. If x and y are both in V_i, $f_i(x, y) = 0$, and $f_i(y, z)$ and $f_i(x, z)$ are both 0 or both 1, according as z lies in V_i or in $Z - V_i$. If x lies in V_i and y in

$Z - V_i$, $f_i(x,y) = 1$. Thus in every case $f_i(x,y) + f_i(y,z) \geqq f_i(x,z)$ for each i, and so $xy + yz \geqq xz$. Hence Z is quasi-metric.

Now let a be a topological limit of the sequence $\{x_n\}$. Take any $r > 0$ and k so large that $1/2^k < r$. If a is not in V_i, $f_i(a, x_n) = 0$ for every n. If a lies in V_i, $f_i(a, x_n) = 0$ for every n larger than some n_i by the definition of topological limit. It we now take m as the largest of the integers n_i, for $i \leqq k$, we have $ax_n = \sum_1^\infty f_i(a, x_n)/2^i \leqq \sum_{k+1}^\infty 1/2^i = 1/2^k < r$ for every $n > m$. Hence a is a u-limit of $\{x_n\}$. Conversely, let $ax_n \to 0$. Let V_k be any vicinity containing a. Since $ax_n \to 0$, there is an integer m such that $ax_n < 1/2^k$ for every $n > m$. This means that x_n lies in V_k for every $n > m$, as otherwise $f_k(a, x_n) = 1$ and $ax_n \geqq 1/2^k$. But this is the definition of topological limit.

To show that Z is separable, consider the sequence of vicinities $\{V_i\}$ and let $W_i = Z - V_i$. In each of the sets V_1 and W_1, if not void, choose a point. In each of the sets $V_1 \cdot V_2$, $V_1 \cdot W_2$, $W_1 \cdot V_2$, and $W_1 \cdot W_2$ which is not void, choose a point. Do the same for $V_1 \cdot V_2 \cdot V_3$, $V_1 \cdot V_2 \cdot W_3$, $V_1 \cdot W_2 \cdot V_3$, $W_1 \cdot V_2 \cdot V_3$, $V_1 \cdot W_2 \cdot W_3$, $W_1 \cdot V_2 \cdot W_3$, $W_1 \cdot W_2 \cdot V_3$, and $W_1 \cdot W_2 \cdot W_3$; etc. At the k-th stage we add at most 2^k new points; hence the set E of these point is an enumerable set.

Now let a be any point of Z. If a lies in V_1, let x_1 be a point of $E \cdot V_1$; if a lies in W_1, let x_1 be a point of $E \cdot W_1$. Also a lies in one of the four sets $V_1 \cdot V_2$, etc.; let x_2 be a point of E in the same set. Let x_3 be a point of E in that one of the eight sets $V_1 \cdot V_2 \cdot V_3$, etc., which contains a; etc. This method of choice insures that a and all the points $\{x_n\}$ lie either in V_1 or in W_1. Hence for every n, $f_1(a, x_n) = f_1(x_n, a) = 0$; and so for every n both ax_n and $x_n a$ are less than 1. Continuing, we see that for any integer m and $n \geqq m$, $f_i(a, x_n) = f_i(x_n, a) = 0$ for $i \leqq m$ and so both ax_n and $x_n a$ are less than $1/2^{m-1}$. Therefore a is the c-limit of the sequence $\{x_n\}$ and Z is separable.

THEOREM IV. *Let Z be a topological space satisfying Axioms A, B, C, 4, and 10. Then distances can be defined so that Z is quasi-metric and separable and, if a is a topological limit of the sequence $\{x_i\}$, then a is an l-limit of the sequence, and conversely.*

To prove this set $f_i(x,y) = 1$ if y lies in V_i and x in $Z - V_i$; otherwise set $f_i(x,y) = 0$. Then proceed much as in the proof of Theorem III.

THEOREM V. *In Theorem III or IV let Z satisfy Axiom 5 as well as 4. Then the same conclusions are valid and also Z as a quasi-metric space satisfies Axiom III′ or III″, respectively.*

Proof. The first statement is true, since Axiom 5 is stronger than Axiom 4. If a and b were both u-limits or l-limits, respectively, of the same sequence $\{x_i\}$, they would both be the topological limits of this sequence by Theorem III or IV, respectively. This, however, is impossible by virtue of Axiom 5. Hence Axiom III′ or III″, respectively, is valid.

THEOREM VI. *Let Z be quasi-metric and separable and satisfy Axiom III′ or III″. Let u-spheres or l-spheres, respectively, be taken for vicinities as in Theorem II. Then Z is a topological space satisfying Axioms A, B, C, 5, and* 10. *The u-limit or l-limit, respectively, of any sequence is the topological limit of the sequence, and vice versa.*

Proof. By Theorem II we need only to prove that Axiom 5 is satisfied. Let a and b be any two points and Axiom III′ be satisfied. Then there is an $r > 0$ such that $U_r(a)$ and $U_r(b)$ have no common points. But by Theorem II these are topological regions and hence Axiom 5 is satisfied. The proof for Axiom III″ is similar.

THEOREM VII. *Theorems V and VI are valid if III′ and III″ are replaced by IV′ and IV″, respectively, and* 5 *by* 6.

Proof. Let us take the u-case and assume Axiom IV′. Let C be u-closed and a be a point not in C. Then there is a $k > 0$ such that $ax > k$ if x lies in C. By Axiom IV′ there is an $r > 0$ and for each point x of C, an $r' > 0$ such that $U_r(a) \cdot U_{r'}(x) = 0$. If $R = U_r(a)$ and S is the union of the sets $U_{r'}(x)$ as x runs over C, R and S are u-regions containing a and C, respectively, and $R \cdot S = 0$. By Theorem II, R and S are topological regions, and hence Axiom 6 is valid.

Conversely, let B be the set of points $\{x\}$ for which $ax \geqq k$. Then B is closed and by Axiom 6 there are disjoint topological regions R and S containing a and B, respectively. By Theorem III there are u-regions; hence some $U_r(a) \subset R$ and, for each x in R, some $U_{r''}(x) \subset S$. Thus Axiom IV′ is satisfied.

8. It has been shown by the researches of Urysohn and Tychonoff * that topological spaces satisfying Axioms A, B, C, 6, and 10 are identical with separable metric spaces. The theorems of the previous section show that, if Axiom 6 is replaced by the weaker Axiom 4 or 5, the resulting topological

* P. Urysohn, "Zum Matrisationsproblem," *Mathematische Annalen*, Vol. 94, pp. 309-315, and A. Tychonoff, "Über einen Metrisationssatz von P. Urysohn," *ibid.*, Vol. 95, pp. 139-142.

spaces can be identified with separable quasi-metric spaces, which in the latter case satisfy Axiom III′ or III″. Theorem VII has been added for the sake of completeness and to show the correspondence between the topological axioms of separation 4, 5, and 6 and the quasi-metrical axioms I, III′ or III″, and IV′ or IV″. Whether this correspondence can be pushed further depends upon the possibility of a converse to Theorem I.

In connection with the above references it should be noted that Theorem II of § 3 as extended at the close of § 5 is Tychonoff's separation theorem (*loc. cit.*, p. 140) and can be proved in a similar manner by the aid of Axiom IV′. Urysohn's method can then be applied to transform a separable quasi-metric space satisfying Axiom IV′ into a separable metric space. For, let $Z = \{x\}$ be the space in question and $E = \{c_i\}$ be the enumerable set dense in Z. For each rational number k and each c_i there is an $r_i > 0$ such that $U_{r_i}(c_i) \subset U_k(c_i)$ and the u-closed sets $U_{r_i}'(c_i)$ and $Z - U_k(c_i)$ have no common points. We can then set up Urysohn's continuous function $f(x)$ and his distance formula just as he does (*loc. cit.*, pp. 311-312).

The following example indicates the possible utility of quasi-metric notions in the study of decompositions of spaces into disjoint sub-sets. Let M be a compact metric space, $M = \sum [X]$ be any decomposition of M into disjoint closed sets, Z be the aggregate whose elements are the sets $\{X\}$, and distances be defined as in the example in § 1. Then Z is quasi-metric. Since M is separable, there is an enumerable set E dense in M. If F denotes any finite sub-set of E, the system of possible sets $\{F\}$ is enumerable. For each F and each rational number $r > 0$ select, if possible, an element Y of Z such that $FY < r$ and $YF < r$; this gives a finite or enumerable aggregate $G = \{Y_i\}$. But by the Borel theorem we see at once that for any $r > 0$ and each X there is some F such that $FX < r$ and $XF < r$. Hence G is dense in Z and Z is separable. This shows incidentally that for any decomposition of a compact metric space into closed sets $\{X\}$ there is an enumerable system G of these sets such that every X is the closed limiting set of some sequence chosen from G. The space Z also satisfies Axiom III′. For, in consequence of our distance definitions, $AX_i \to 0$ if and only if the upper closed limiting set of $\{X_i\}$ is a sub-set of A. As the elements of Z, considered as sub-sets of M, are mutually disjoint, no sequence of elements of Z has more than one u-limit. As a topological space Z satisfies Axioms A, B, C, 5, and 10.

YALE UNIVERSITY,
NEW HAVEN, CONN.

DOUBLE IMPLICATION AND BEYOND.

By H. B. Smith.

The general problem which we have here in mind and which will only be begun in the present paper but continued in a later one, is to determine (to construct) all the true and all the untrue propositions into which any number of symbols of implication may enter in any way whatsoever. The word "true" is to be understood as "true in all senses of the word true," whereas the word "untrue" is to be taken to mean "untrue in at least one sense of the word," that is to say, not generally true. The construction of an infinite series of meanings of the true and the untrue and their definitions as well as a statement of the elementary properties of the existential function is contained in an article in the *Bulletin of the American Mathematicol Society,* Jan.-Feb. 1929. This function is defined by

$$|XY|' = X \angle Y'$$

which may be read: If X (is true) then Y (is untrue). Our fundamental assumption is,

$$X \angle |X|$$

that is: If X (is true) then X (is true for some meanings of the terms that enter into X). The converse is in general untrue. The *degree* of the function is the number of its elements or variables. Its *order* is the largest number of operations that occur among its terms when the function is expanded. Thus $|QR|$ is first order, $|P|QR||$ is second order and so on.

Since every proposition into which any number of symbols of implication enters in any way undetermined, may be expressed as a function of existentials of varying orders and degrees and of the free variables, the problem may be otherwise stated: To determine all the cases in which ϕ becomes unity, where ϕ is a function of any form whatever of existentials of undetermined order and degree and of the free variables.

Without loss of generality we may suppose the symbol of implication to appear nowhere explicitly in ϕ, being implicit in the existentials, and that the variables in turn, unless otherwise stated, are free variables, that is free of the symbol of implication. If we say that ϕ and the variables entering into ϕ are general, we mean that they may represent either sums of products or products of sums. This latter provision may in turn be dropped since a product of sums may always be represented as a sum of products by direct

multiplication. It may be supposed, then, that ϕ is of the form of a sum of products and the question then is, under what conditions does ϕ become unity.

The problem can again be stated otherwise and in a way that will lead more directly to a solution, for if ϕ is primarily a sum of products, ϕ' is in turn reducible to a sum of products. The condition for ϕ becoming unity will be the same as for the vanishing of ϕ', and since for a sum to vanish each term must vanish separately, our problem may now be stated in its simplest and most general form: To determine all the cases in which ϕ vanishes (replacing ϕ' by ϕ) where ϕ is a product of existentials of any order and degree and the free variables, the latter being in general sums of products of the categorical forms.

Let us begin with the consideration of some simple cases. The double implication,

$$P \angle (Q \angle R') = | P | QR | |'$$

is true when and only when $P | QR |$ vanishes. This will happen if P or Q or R or QR implies zero but we shall find cases in which

$$P | QR | \angle o$$

holds without any of these conditions being satisfied. Again the triple implication,

$$P \angle [Q \angle (R \angle S')] = | P | Q | RS | | |'$$

will only be true when $P | Q | RS | |$ vanishes, which will occur if any of the variables imply zero or again if

$$Q | RS | \angle o.$$

These last cases having been determined we should still have to consider if there are instances in which

$$P | Q | RS | | \angle o$$

without the aforementioned conditions being fulfilled. Here our task is somewhat simplified by the fact that

$$Q | RS | \angle o$$

is not true unless

$$QRS \angle o$$

that the second order expression

$$P | Q | RS | | \angle o$$

does not hold except for the cases in which

$$PQ \mid RS \mid \angle o$$

holds true, and so on. This is because QRS is a term in the expansion of $Q \mid RS \mid$, $PQ \mid RS \mid$ is a term in the expansion of $P \mid Q \mid RS \mid \mid$ etc.

The other double implication,

$$(P \angle Q') \angle R' = \mid \mid PQ \mid' R \mid'$$

requires the vanishing of $\mid PQ \mid' R$ and this will come about if $R = o$ or if $P' = Q' = o$, or if $R = PQ$, but we shall find cases in which

$$\mid PQ \mid' R \angle o$$

holds independently of these conditions. Again the triple implication,

$$[(P \angle Q') \angle R'] \angle S' \qquad \mid \mid PQ \mid' R \mid' S \mid'$$

is satisfied if $S = \mid PQ \mid' R$ or if

$$(1)\ S = o \quad \text{or} \quad (2)\ R' = o \quad \text{and} \quad PQ = o$$

and, in the general case if,

$$(1)\ T = o \quad \text{or} \quad (2)\ S' = o \quad \text{and} \quad \cdots R' = o \quad \text{and} \quad PQ = o.$$

The other possibilities, however, remain and the solution of the general case will have to be built up inductively from the simple to the more complex.

The traditional categorical forms we shall represent by the notation:

$$\begin{aligned} A(ab) &= \text{All } a \text{ is } b \\ E(ab) &= \text{No } a \text{ is } b \\ I(ab) &= \text{Some } a \text{ is } b \\ O(ab) &= \text{Not all } a \text{ is } b \end{aligned}$$

the term-order being the order subject-predicate. Whenever it is desired to indicate that the term-order is unsettled, a comma will appear between the terms. The set of propositions of a given type will be called the *array* of propositions of that type. Let us consider at the outset the array,

$$\mid X(a, b) \mid' \mid Y(a, b) \mid' \angle o$$

wherein X and Y are conceived as capable of taking on any of the values A, E, I, O and in which

$$\mid X(a, b) \mid' = X(a, b) \angle o.$$

There will evidently be sixteen propositions in the array obtained by taking the permutations of the four letters two at a time and by taking each letter once with itself. We shall refer to the case in which the term-order is the same in X and Y as the first *figure* of the array, to the case in which the term-order in X is the reverse of the term-order in Y as the second figure. There will then be thirty-two instances to consider. The true members of the set will be called valid *moods*, the others invalid. The valid moods, AO, EI, IE, IO, OA, OI in the first figure follow at once from $A'O' \angle o$, $E'I' \angle o$, $I'O' \angle o$, results of single implication (see the writer's *Symbolic Logic*, Chap. III, Crofts, N. Y.) by "strengthening," that is, by the principle,*

$$(XY \angle Z)(W \angle X) \angle (WY \angle Z)$$

for since $X(ab) \angle \mid X(ab) \mid$ and therefore $\mid X(ab) \mid' \angle X'(ab)$, we have

$$\{A'(ab)O'(ab) \angle o\}\{\mid A(ab)\mid' \angle A'(ab)\} \angle \{\mid A(ab)\mid' O'(ab) \angle o\}$$
$$\{\mid A(ab)\mid' O'(ab) \angle o\}\{\mid O(ab)\mid' \angle O'(ab)\} \angle \{\mid A(ab)\mid' \mid O(ab)\mid' \angle o\}$$

and the valid moods of the second figure, EI, IE, IO, OI, follow from those of the first by converting simply, that is by interchanging terms in an E or I form. This process could of course be represented symbolically. The invalid moods composed of two affirmative forms, A and I, can be shown to be invalid by the substitution $a = b'$, those composed of two negative forms, E and O, by the substitution $a = b$.† The rest are theorems from the

Postulate: $\mid A(ab) \mid' \mid O(ba) \mid' \angle o$ is untrue.

We have

$$\begin{aligned} \{X \angle Y\} &\angle \{\mid X \mid \angle \mid Y \mid\} \\ &\angle \{\mid \mid Y \mid' \mid \angle \mid \mid X \mid' \mid\} \\ &\angle \{\mid \mid X \mid' \mid' \angle \mid \mid Y \mid' \mid'\} \\ \{X \angle Y\} &\angle \{[U \mid X \mid' \angle o] \angle [U \mid Y \mid' \angle o]\} \end{aligned}$$

and accordingly,

$$\{\mid A(ab) \mid' \mid O(ba) \mid' \angle o\}' \angle \{\mid A(ab) \mid' \mid E(ba) \mid' \angle o\}'.$$

The rest follow by converting simply in the E-form. We may note in passing

---

* The principles which we take from the calculus of propositions, may, however, be derived by assuming certain very simple properties of the existential function together with the formulas for its expansion.

† For a proof that the affirmative forms become true, the negative forms false, when subject and predicate are identified, that the negative forms become true, the affirmative forms false, when the terms are made contradictory, see *Symbolic Logic*, p. 77.

that this assumption enables us to save the postulate we were forced to introduce in *Symbolic Logic,* page 80.

There are no valid moods in the array,

$$| X(a,b)Y(b,c) |' \, | Z(c,a) |' \angle o$$

for if X or Y is negative we have only to identify terms in the negative, if X and Y are affirmative and Z negative to identify terms in Z, if X, Y and Z are affirmative to identify terms in X or Y, in order to make the invalidity of the mood depend on the invalidity of a simpler case.

The array $X(a,b) \mid Y(a,b) |' \angle o$.

The valid moods follow at once from the valid moods of immediate inference by "strengthening," thus,

$$\{O(ab)O'(ab) \angle o\}\{ \mid O(ab) |' \angle O'(ab)\} \angle \{O(ab) \mid O(ab) |' \angle o\}$$
$$\{A(ab)\, I'(ba) \angle o\}\{ \mid I(ba) |' \angle I'(ba)\} \angle \{A(ab) \mid I(ba) |' \angle o\} \text{*}$$

The invalid moods are deduced by the methods illustrated below.

$$\{ \mid A(ab)|' \angle A'(ab)\}\{ \mid A(ab)|' \mid O(ba)|' \angle o\}' \angle \{A'(ab) | O(ba)|' \angle o\}'$$
by $(XY \angle Z)'(X \angle W) \angle (WY \angle Z)'$

$$\{A'(ab) \angle O(ab)\}\{A'(ab) \mid O(ba) |' \angle o\}' \angle \{O(ab) \mid O(ba) |' \angle o\}'$$
by the same principle.

$$\{E(ab) \angle O(ba)\} \angle \{[O(ab) \mid O(ba) |' \angle o]' \angle [O(ab) \mid E(ab) |' \angle o]'\}$$
by $\{X \angle Y\} \angle \{[U \mid Y |' \angle o]' \angle [U \mid X |' \angle o]'\}$.

If $A(ab) \mid E(ab) |' \angle o$ were valid it would have to remain valid for all special meanings of the terms. Substitute $a = b$. The antecedent then becomes true and the consequent false.

It might be well to indicate at this point a method alternative to the one we have employed so far for the determination of the invalid moods.

Postulate: There exists a meaning of a and b, viz. $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $A(\boldsymbol{ab}) \angle o$ and $O(\boldsymbol{ba}) \angle o$.

These meanings might be regarded as empirical or definitional (from some other science), for example *plane figures* and *triangles.* Our derivations would then proceed as follows:

THEOREM. $| A(\boldsymbol{ab}) |' \, | O(\boldsymbol{ba}) |' \angle o$ *is untrue,*

since it is the form $i \angle o$ or zero, and from this,

---

* **For a proof that the current view that subalternation fails rests on a misunderstanding see *Symbolic Logic,* Chap. III.**

THEOREM. $| A(ab) |' \, | O(ba) |' \angle o$ *is untrue,*

since it is true for *not all* meanings of the terms, and accordingly,

$$\{ \, | A(\boldsymbol{ab}) |' \, | O(\boldsymbol{ba}) |' \angle o \}' \angle \{ \, | A(\mathbf{ab}) |' \, | E(\mathbf{ba}) |' \angle o \}'.$$

The antecedent *being true may be suppressed* and we have,

THEOREM. $| A(\boldsymbol{ab}) |' \, | E(\boldsymbol{ba}) |' \angle o$ *is untrue,*

THEOREM. $| A(ab) |' \, | E(ba) |' \angle o$ *is untrue,*

since it is true for *not all* meanings of the terms.

Similarly, in the case of the assumption we introduce below, we might write

Postulate: There exists a meaning of a, b and c, viz. $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, such that $O(\boldsymbol{ab}) \angle o$, $O(\boldsymbol{cb}) \angle o$ and $I(\boldsymbol{ca}) \angle o$.

THEOREM. $| O(\boldsymbol{ab}) |' \, | O(\boldsymbol{cb}) |' \, | I(\boldsymbol{ca}) |' \angle o$ *is untrue,*

THEOREM. $| O(ab) |' \, | O(cb) |' \, | I(ca) |' \angle o$ *is untrue,*

$$\{ \, | O(\boldsymbol{ab}) |' \, | O(\boldsymbol{cb}) |' \, | I(\boldsymbol{ca}) |' \angle o \}' \angle \{ \, | E(\boldsymbol{ab}) |' \, | O(\boldsymbol{cb}) |' \, | I(\boldsymbol{ca}) |' \angle o \}'$$
$$\text{by } \{X \angle Y\} \angle \{[U \, | X |' \angle o] \angle [U \, | Y |' \angle o]\}.$$

THEOREM. $| E(\boldsymbol{ab}) |' \, | O(\boldsymbol{cb}) |' \, | I(\boldsymbol{ca}) |' \angle o$ *is untrue,*

THEOREM. $| E(ab) |' \, | O(cb) |' \, | I(ca) |' \angle o$ *is untrue*

and so on. This method does not apply, however, to the later postulates we shall set down.

The array $| X(a, b) |' \, | Y(b, c) |' \, | Z(c, a) |' \angle o$.

We should have here sixty-four cases to consider, each one in each one of the ordinary four figures, though because logical multiplication is commutative not all of these will be distinct. The valid moods are obtained by "strengthening" the premises in $X'(a, b) \; Y'(b, c) \; Z'(c, a) \angle o$. Thus from BARBARA by three steps:

$$\{O'(ba) \; O'(cb) \; A'(ca) \angle o\}\{| O(ba) |' \angle O'(ba)\} \angle \{| O(ba) |' \, O'(cb) \; A'(ca) \angle o\}$$
$$\{| O(ba) |' \, O'(cb) \; A'(ca) \angle o\}\{| O(cb) |' \angle O'(cb)\} \angle \{| O(ba) |' \, | O(cb) |' \, A'(ca) \angle o\}$$
$$\{| O(ba) |' \, | O(cb) |' \, A'(ca) \angle o\}\{| A(ca) |' \angle A'(ca)\} \angle \{| O(ba) |' \, | O(cb) |' \, | A(ca) |' \angle o\}$$

For the deduction of the invalid moods we shall introduce a

Postulate: $| O(ab) |' \, | O(cb) |' \, | I(ca) |' \angle o$ is untrue.

From this assumption and by the methods exemplified below all the invalid moods can be established. We may note in passing that we are now able to dispense with the postulate introduced in *Symbolic Logic,* page 98.

(1) Suppose $|A(ba)|'|A(cb)|'|A(ca)|' \angle o$ were valid and make $b = a'$. The mood then reduces to $|E(ca)|'|A(ca)|' \angle o$ a form already established as invalid.*

(2) Suppose $|A(ab)|'|O(bc)|'|O(ca)|' \angle o$ were valid and make $c = a$. The mood then reduces to $|A(ab)|'|O(ba)|' \angle o$.

(3) $\{|O(ab)|'|O(cb)|'|I(ca)|' \angle o\}' \angle \{|E(ab)|'|O(cb)|'|I(ca)|' \angle o\}'$
by $\{X \angle Y\} \angle \{[U|X|' \angle o] \angle [U|Y|' \angle o]\}$.

There are no valid moods in the array,

$$|X(a,b)\,Y(b,c)|'|Z(c,d)|'|W(d,a)|' \angle o.$$

In order to show this identify terms in a negative form if a negative form occurs. If all the forms are affirmative, we have only to identify terms in one of the forms of the first bracket.

The array $X(a,b)\,Y(b,c)\,|Z(c,a)|' \angle o$.

The valid moods are derivable from and correspond exactly to the valid moods of the syllogism. The invalid moods are gotten from the invalid moods of the last array. The following examples will be enough to illustrate the method.

(1) $\{A(ba)A(cb)A'(ca) \angle o\}\{|A(ca)|' \angle A'(ca)\}$
$\angle \{A(ba)A(cb)\,|A(ca)|' \angle o\}$

(2) $\{E(ca)\,|O(ca)|' \angle o\}\{A(ab)E(cb) \angle E(ca)\}$
$\angle \{A(ab)E(cb)\,|O(ca)|' \angle o\}$
by $(XY \angle Z)(W \angle X) \angle (WY \angle Z)$

(3) $\{|O(ab)|'|O(cb)|'|I(ca)|' \angle o\}'\{|O(ab)|' \angle O'(ab)\}$
$\angle \{O'(ab)|O(cb)|'|I(ca)|' \angle o\}'$
$\{O'(ab)\,|O(cb)|'|I(ca)|' \angle o\}'\{|O(cb)|' \angle O'(cb)\}$
$\angle \{O'(ab)O'(cb)\,|I(ca)|' \angle o\}'$
$\{O'(ab)O'(cb)\,|I(ca)|' \angle o\}'\{O'(ab) \angle A(ab)\}$
$\angle \{A(ab)O'(cb)\,|I(ca)|' \angle o\}'$
$\{A(ab)O'(cb)\,|I(ca)|' \angle o\}'\{O'(cb) \angle A(cb)\}$
$\angle \{A(ab)A(cb)\,|I(ca)|' \angle o\}'$
by $(XY \angle Z)'(X \angle W) \angle (WY \angle Z)'$

* For a proof of obversion see *Symbolic Logic,* page 76.

The array $X(a, b) \mid Y(b, c) \mid' \mid Z(c, a) \mid' \angle o$.

There is no novel principle involved in this case. The valid moods come from the array of the syllogism $X(a, b)\ Y'(b, c)\ Z'(c, a) \angle o$ by "strengthening," the invalid moods from the array $\mid X(a, b) \mid' \mid Y)b, c) \mid' \mid Z(c, a) \mid' \angle o$ by "weakening" as before.

The array $\mid X(1, 2) \mid' \mid Y(2, 3) \mid' \cdots \mid Z(n, 1) \mid' \angle o$

wherein the n terms are arranged in a cycle and the number of premises is the same as the number of terms. All valid moods of this type are evidently gotten from valid moods of the sorites as those of the cycle of three terms are gotten from valid moods of the cycle of three terms or the syllogism. These types already established (*Symbolic Logic,* Chap. V) are:

$$O'(21)\ O'(32) \cdots O'(n\, n-1) \angle O'(n\, 1)$$
$$O'(21)\ O'(32) \cdots O'(r\, r-1)\ O'(r\, r+1) \cdots$$
$$O'(n-2\, n--1)\ O'(n-1\, n) \angle E'(n\, 1)$$
$$O'(21)\ O'(32) \cdots O'(t\, t-1)\ E'(t, t+1)\ O'(t+1\, t+2) \cdots$$
$$O'(n-2\, n-1)\ O'(n-1\, n) \angle E'(n\, 1).$$

The invalidity of all other moods can be established by the same methods (*Symbolic Logic,* Chap. V) as before and it will be easy to show that no bracket can contain more than one form of the cycle. More generally, valid moods in which only some existentials occur correspond exactly to valid moods of the sorites wherein any $Y'(r, r-1)$ is strengthened to $\mid Y(r, r-1) \mid'$.

It may be useful to point out one method of establishing invalidity by reduction to a special case, a method not needed and therefore not employed in *Symbolic Logic,* Chap. V. Suppose we are dealing with an implication of the form,

$$X(1, 2)\ Y(2, 3) \cdots Z(n-1, n) \mid U(n, n+1) \cdots V(r\, 1) \mid' \angle o.$$

If there is more than one negative form in the second part, eliminate the affirmative forms through the identification of terms. If Z and U be adjacent, that is, if they have a term in common, make $n + 1 = n'$ and get rid of the $(n + 1)'$ in Z by obversion, and continue thus to eliminate the forms in the second bracket.

Zero Conjunction of Cycles, Chains and Zero Conjunction of Chains.

Two categorical forms which have a common term are called *adjacent.* A product of categorical forms and of existentials of these products wherein

each form is adjacent to at least one other form is called a *chain.* Thus a *cycle,* a conjunction of cycles or a series of cycles one-directionally joined is a special case of a chain closed at some of its terms. The terms common to two chains are called their *intersections.* The cycles formed at the intersections in a product of two or more chains are called their *cross-products.* A chain of forms represented as implying zero is called a *zero chain.* A product of chains represented as implying zero is called a *zero conjunction of chains.*

If $P \mid Q \mid' \angle\ o$ be of such a form that PQ represents a zero conjunction of cycles, it being understood that existentials may appear as factors, then Q contains among its factors no zero cycle and no cross-product that vanishes unless P also vanishes, for in that case $\mid Q \mid'$ reduces to unity. We may suppose, then, that if PQ contains any vanishing cycle, that its factors lie partly in P and partly in Q. Moreover there are no terms among the categorical forms that appear in Q that do not also appear in P. For if any term in Q did not appear in P we could identify it with an adjacent term in the case of a negative, make it contradictory with its adjacent term in the case of an affirmative, and so cause Q to vanish. The invalidity of the mood would then be establshed since P is assumed not to vanish independently.

Consider now the implication,

$$C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle\ o$$

in which each C is in the form of a cycle of any number of terms. The ordinary zero conjunction of cycles of single implication if valid contains a zero factor, either one of the C's or some cross product. Let this factor be C and its form,

$$X'(1, 2)\ Y'(2, 3) \cdot \cdot \cdot Z'(n\ 1) \angle\ o.$$

By strengthening each $W'(r, r-1)$ to $\mid W(r, r-1) \mid'$ in succession there would arise corresponding valid moods of

$$C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle\ o$$

Principle: If none of the cycles of a valid mood of the zero conjunction of cycles vanishes, then one of the cross-products vanishes.

From this principle all valid moods are evidently constructible, for if K be the cross-product in question,

$$(K \angle\ o) \angle (C_1\, C_2\, C_3 \cdot \cdot \cdot C_n \angle\ o)$$

Principle: A valid mood of the zero chain contains at least one vanishing cycle.

Principle: A valid mood of the zero conjunction of chains contains at least one vanishing cycle.

These principles, of which the first two are special cases of the last, evidently contain the general solution of $P \mid Q \mid' \angle\, o$ for we have now constructed all valid implications of this type, P and Q being in general sums of products of the categorical forms. If

$$P = lm \cdots + pq \cdots + uv \cdots + \cdots$$
$$\mid Q \mid' = \mid n + r + w + \cdots + S \mid' = \mid n \mid' \mid r \mid' \mid w \mid' \cdots \mid S \mid'$$

the valid moods would be generally those in which each one of the factors, say,

$$l \mid n \mid', \qquad p \mid r \mid', \qquad u \mid w \mid',$$

vanishes.

The array $X(a, b) \mid Y(a, b) \mid \angle\, o$.

The valid moods are gotten from a postulate we shall introduce later on, viz. $\mid A(ab) \mid \mid E(ab) \mid \angle\, o$, as follows:

$$\{ \mid A(ab) \mid \mid E(ab) \mid \angle\, o\}\{A(ab) \angle \mid A(ab) \mid \} \angle \{A(ab) \mid E(ab) \mid \angle\, o\}$$
$$\{ \mid E(ab) \mid \mid A(ab) \mid \angle\, o\}\{E(ab) \angle \mid E(ab) \mid \} \angle \{E(ab) \mid A(ab) \mid \angle\, o\}$$
$$\text{by } (XY \angle Z)(W \angle X) \angle (WY \angle Z).$$

We derive the invalid moods in part from two later postulates, viz.

$$\mid E(ba)\, A(bc) \mid A(ca) \angle\, o \quad \text{and} \quad E(ba)\, A(bc) \mid A(ca) \mid \angle\, o$$

and a postulate already introduced,

$$\mid A(ab) \mid' \mid O(ba) \mid' \angle\, o$$

all of these being assumptions of invalidity, as follows:

$$\{ \mid E(ba)A(bc) \mid A(ca) \angle\, o\}'\{ \mid E(ba)A(bc) \mid \angle \mid O(ca) \mid \} \angle \{ \mid O(ca) \mid A(ca) \angle\, o\}'$$
$$\{E(ba)A(bc) \mid A(ca) \mid \angle\, o\}'\{E(ba)A(bc) \angle O(ca)\} \angle \{O(ca) \mid A(ca) \mid \angle\, o\}'$$
$$\text{by } (XY \angle Z)' (X \angle W) \angle (WY \angle Z)'$$
$$\{ \mid A(ab) \mid' \mid O(ba) \mid' \angle\, o\}' \{ \mid A(ab) \mid' \angle \mid O(ab) \mid \} \angle \{ \mid O(ab) \mid \mid O(ba) \mid' \angle\, o\}'$$
$$\{ \mid O(ab) \mid \mid O(ba) \mid' \angle\, o\}' \{ \mid O(ba) \mid' \angle O'(ba)\} \angle \{ \mid O(ab) \mid O'(ba) \angle\, o\}'.$$

This is $A(ba) \mid O(ab) \mid \angle\, o$ is untrue, and in the same way we obtain $O(ba) \mid A(ab) \mid \angle\, o$ is untrue, while $E(ab) \mid I(ab) \mid \angle\, o$ is untrue and $\mid E(ab) \mid I(ab) \angle\, o$ is untrue follow from these by obversion. Again the invalidity of $I(ab) \mid O(ab) \mid \angle\, o$ and $\mid I(ab) \mid O(ab) \angle\, o$ are derived from the same theorems by weakening and the remainder can be established by one the substitutions $a = b$ or $a = b'$.

The array $|X(a,b)|\,|Y(a,b)| \angle o$.

There is only one valid mood in this set and this we shall introduce provisionally as a

Postulate: $|A(ab)|\,|E(ab)| \angle o$.

It is a matter of indifference whether we assume this or the two that follow from it in the array that has just gone before, for these give together

$$|A(ab)|\,|E(ab)| \angle A'(ab)E'(ab)$$

or $$|A(ab)|\,|E(ab)| \angle A'(ab)E'(ab)\ |A(ab)|\,|E(ab)|$$

and this last vanishes, since it is a term in the expansion of

$$|A(ab)E(ab)| \angle o$$

by the fundamental formula introduced elsewhere (*Symbolic Logic*, p. 127)

$$|XY| = XY + X'Y'\,|X|\,|Y| + |X|\,|Y'|' + |Y|\,|X'|'.$$

All the invalid moods follow from those that have been established in the array that has gone before by weakening.

It might be well before we leave these arrays of two-term cycles to give at least one illustration of the fallacy of assuming

$$X \angle Y = X' + Y.$$

Thus, $$X \angle (Y \angle Z) = X' + (Y \angle Z) = X' + Y' + Z = XY \angle Z$$

$$\{A(ba)A(cb) \angle A(ca)\} \angle \{A(ba) \angle [A(cb) \angle A(ca)]\}$$

gives for $b = c$,

$$A(ba) \angle |A'(ba)|' = A(ba) \angle |O(ba)|' = |O(ba)| \angle O(ba)$$

which is fallacious.

The array $|X(a,b)Y(b,c)|\,Z(c,a) \angle o$.

The valid moods, six in number, are derived from those of the last array but one by strengthening as follows:

$$\{|A(ca)|\,E(ca) \angle o\}\{|A(ba)A(cb)| \angle |A(ca)|\} \angle \{|A(ba)A(cb)|\,E(ca) \angle o\}$$
$$\{|E(ca)|\,A(ca) \angle o\}\{|E(ab)A(cb)| \angle |E(ca)|\} \angle \{|E(ab)A(cb)|\,A(ca) \angle o\}.$$

For the derivation of the invalid moods we shall introduce

Postulate: $|A(ba)A(bc)|\,E(ca) \angle o$ *is untrue,*

THEOREM: $|E(ba)A(bc)|\,A(ca) \angle o$ *is untrue,*

by making $a = a'$.

One illustration will serve to indicate the method by which many theorems can be derived:

$$\{A(ab)A(cb)E(ca) \angle o\}' \{A(ab)A(cb) \angle \mid A(ab)A(cb) \mid \} \\ \angle \{ \mid A(ab)A(cb) \mid E(ca) \angle o\}'.$$

When this method is not applicable the invalidity of the mood can be established by one of the substitutions, $a = b, b', \; c = b, b'$.

The array $X(a, b)Y(b, c) \mid Z(c, a) \mid \angle o$.

Here the derivations are entirely analogous to the last case. We have:

Postulate: $A(ba)A(bc) \mid E(ca) \mid \angle o$ is untrue,

THEOREM: $E(ba)A(bc) \mid A(ca) \mid \angle o$ is untrue,

and the procedure is the same as before. In passing, however, we may note again one case of independent interest:

$$\{ \mid A(ba)A(bc) \mid E(ca) \angle o\}' \{ \mid A(ba)A(bc) \mid \angle \mid I(ca) \mid \} \\ \angle \{I(ca) \mid E(ca) \mid \angle o\}'$$
$$\{A(ba)A(bc) \mid E(ca) \mid \angle o\}' \{A(ba)A(bc) \angle I(ca)\} \\ \angle \{E(ca) \mid I(ca) \mid \angle o\}'$$

by the principle that we have repeatedly used before.

The valid moods of the zero cycle are gotten at once by strengthening the factors in

$$A(n1) \mid E(n1) \mid \angle o, \qquad E(n1) \mid A(n1) \mid \angle o,$$

by means of the implications,

$$A(12) \cdots A(t-1\,t)E(t, t+1)A(t+2\,t+1) \cdots A(n\,n-1) \angle E(n1)$$
$$A(21)A(32) \cdots A(n-1\,n-2)A(n\,n-1) \angle A(n1)$$

giving rise to four types. All the others can be shown to be invalid by the methods developed already (*Symbolic Logic,* Chap. V). If now we introduce again the principles laid down before, the valid moods of the zero chain, the zero conjunction of cycles and the zero conjunction of chains will be determined. We have then arrived at the general solution of $P \mid Q \mid \angle o$, P and Q being perfectly general, that is, variables of any form free of the symbol of implication.

UNIVERSITY OF PENNSYLVANIA.

REGULAR BILINEAR TRANSFORMATIONS OF SEQUENCES.

By P. A. Fraleigh.

Introduction. Linear transformations of sequences of the form

$$S: \qquad y_n = \sum_{k=1}^{n} a_{nk} x_k,$$

where (x_k) is a given sequence and a_{nk} is constant, have been used * to evaluate divergent series. Such a transformation is said to be *regular* if $\lim_{n\to\infty} y_n$ exists and equals $\lim_{n\to\infty} x_n$, whenever the latter exists. The following theorem is due to Silverman and Toeplitz.

Theorem A. *A necessary and sufficient condition that transformation S be regular is that $\lim_{n\to\infty} a_{nk} = 0$, for each k; $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$; $\sum_{k=1}^{n} | a_{nk} | < C$, for all n, where C is a constant.*

Another theorem due to Kojima † is as follows:

Theorem B. *A necessary and sufficient condition that $\lim_{n\to\infty} y_n$ exist, whenever $\lim_{n\to\infty} x_n$ exists, is that $\lim_{n\to\infty} a_{nk}$ exist for each k; $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}$ exist; $\sum_{k=1}^{n} | a_{nk} | < C$, for all n. If these conditions hold then*

$$\lim_{n\to\infty} y_n = \alpha l + \sum_{k=1}^{\infty} \alpha_k (x_k - l),$$

where

$$\alpha = \lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}, \qquad \alpha_k = \lim_{n\to\infty} a_{nk}, \qquad \textit{and} \qquad \lim_{n\to\infty} x_n = l.$$

In this paper we shall be concerned with bilinear transformations of sequences of the form

$$T: \qquad y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} u_k v_l,$$

where (u_k) and (v_l) are sequences and a_{nkl} is constant. Such a transformation is *regular* if $y_n \to uv$ whenever $u_n \to u$ and $v_n \to v$.

---

* First studied by L. L. Silverman, *Missouri Dissertation* (1910), and Toeplitz, *Prace matematyzno-ficyczne*, Vol. 28 (1911), p. 113.

† Tetsuzô Kojima, *Tôhoku Mathematical Journal*, Vol. 12 (1917).

In § 1 we shall give a set of necessary and sufficient conditions for regularity of T. A further necessary condition will be found in § 2. An application will be made, in § 3, to the Cauchy product of two series each of which is Cèsaro summable. A set of necessary and sufficient conditions will be given under which the Cauchy product of two Cèsaro summable series is evaluated correctly by the transformation S. Finally in § 4 there are some remarks concerning the possibility of further simplifying the conditions for regularity.

1. *Necessary and sufficient conditions for regularity.* Given two sequences (u_n) and (v_n), consider the bilinear transformation defined by

$$T: \qquad y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} u_k v_l, \qquad\qquad a_{nkl} \text{ constant.}$$

If $y_n \to uv$, whenever $u_n \to u$ and $v_n \to v$, we say that T is *regular.*

Let
$$V_{nk} = \sum_{l=1}^{n} a_{nkl} v_l.$$
Then
$$y_n = \sum_{k=1}^{n} V_{nk} u_k,$$

and, by Theorem B, a necessary and sufficient condition that $y_n \to uv$ whenever $u_n \to u$ is that $\lim_{n\to\infty} V_{nk} = 0$, for each k; $\lim_{n\to\infty} \sum_{k=1}^{n} V_{nk} = v$; $\sum_{k=1}^{n} | V_{nk} | < C$, for all n, where C is a constant.

A necessary and sufficient condition that $\lim_{n\to\infty} V_{nk} = 0$ for each k, whenever $v_n \to v$, is that $\lim_{n\to\infty} a_{nkl} = 0$, for each k and l; $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ for each k; $\sum_{l=1}^{n} | a_{nkl} | < K(k)$ for each k and all n, where $K(k)$ is a constant depending on k.

We may write
$$\sum_{k=1}^{n} V_{nk} = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl}\right) v_l,$$

so that a necessary and sufficient condition that $\sum_{k=1}^{n} V_{nk} \to v$, whenever $v_n \to v$, is that $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$, for each l; $\lim_{n\to\infty} \sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl} = 1$; $\sum_{l=1}^{n} | \sum_{k=1}^{n} a_{nkl} | < C$, for all n, where C is a constant.

We may write $\sum_{k=1}^{n} | V_{nk} | < C$ in the form

$$\sum_{k=1}^{n} \left| \sum_{l=1}^{n} a_{nkl} v_l \right| < B(v_1, v_2, \cdots),$$

where $B(v_1, v_2, \cdots)$ is a constant for each convergent sequence (v_n). We have, therefore, the following theorem:

THEOREM I. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$, *for each k and l;* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$, *for each l;* $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$, *for each k;* $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$; $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \right| < C$, *for all n;* $\sum_{l=1}^{n} | a_{nkl} | < K(k)$ *for each k and all n;* $\sum_{k=1}^{n} \left| \sum_{l=1}^{n} a_{nkl} v_l \right| < (B(v_1, v_2, \cdots)$ *for all n; where C is a constant, $K(k)$ is a constant for each k, and $B(v_1, v_2, \cdots)$ is a constant for each convergent sequence (v_n).*

By interchanging the rôles of k and l, we obtain the following theorem.

THEOREM I′. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each k and l,* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each l,* $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each k,* $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \left| \sum_{l=1}^{n} a_{nkl} \right| < C$ *for all n,* $\sum_{k=1}^{n} | a_{nkl} | < L(l)$ *for each l and all n,* $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A(u_1, u_2, \cdots)$ *for all n, where C is a constant, $L(l)$ is a constant for each l, and $A(u_1, u_2, \cdots)$ is a constant for each convergent sequence (u_n).*

We now prove certain lemmas which simplify the conditions of these theorems.

LEMMA 1. *From* $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A(u_1, u_2, \cdots)$ *and* $\sum_{l=1}^{n} | a_{nkl} | < K(k)$, *follows* $\sum_{l=1}^{n} | a_{nkl} | < K$ *for all k and n, where K is a constant.*

We will assume that

$$\sum_{l=1}^{n} | a_{nkl} | \geqq K$$

is true for a sequence of pairs of values of n and k and obtain a contradiction.

Let M_k be the greatest of $K(1), K(2), \cdots, K(k)$. Choose n_1 and k_1 such that

$$\sum_{l=1}^{n_1} | a_{n_1 k_1 l} | \geqq 1.$$

Define
$$u_k = 0, \qquad 1 \leqq k \leqq n_1, \qquad k \neq k_1,$$
$$u_{k_1} = 1.$$

Then
$$\sum_{l=1}^{n_1} \left| \sum_{k=1}^{n_1} a_{n_1 k l} u_k \right| = \sum_{l=1}^{n_1} | a_{n_1 k_1 l} | \geqq 1.$$

In general, choose $n_p > n_{p-1}$ and $k_p > n_{p-1}$ such that
$$\sum_{l=1}^{n_p} | a_{n_p k_p l} | \geqq 2^{p-1}(p-1) M_{n_{p-1}} + 2^{2p-1}.$$

Define
$$u_k = 0, \qquad n_{p-1} < k \leqq n_p, \qquad k \neq k_p,$$
$$u_{k_p} = \frac{1}{2^{p-1}}.$$

Then
$$\left| \sum_{k=1}^{n_p} a_{n_p k l} u_k \right| = \left| a_{n_p k_1 l} + \tfrac{1}{2} a_{n_p k_2 l} + \cdots + \frac{1}{2^{p-1}} a_{n_p k_p l} \right|,$$
and
$$\sum_{l=1}^{n_p} \left| \sum_{k=1}^{n_p} a_{n_p k l} u_k \right| \geqq \frac{1}{2^{p-1}} \sum_{l=1}^{n_p} | a_{n_p k_p l} | - \frac{1}{2^{p-2}} \sum_{l=1}^{n_p} | a_{n_p k_{p-1} l} | - \cdots - \sum_{l=1}^{n_p} | a_{n_p k_1 l} |$$
$$\geqq \frac{1}{2^{p-1}} [2^{p-1}(p-1) M_{n_{p-1}} + 2^{2p-1}] - \frac{K(k_{p-1})}{2^{p-1}} - \cdots - K(k_1)$$
$$\geqq 2^p.$$

For this particular sequence $u_n \to 0$, while the expression
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right|$$
is not bounded. The condition of lemma 1 is therefore necessary.

LEMMA 2. *When the first six conditions of Theorem I are satisfied, a necessary and sufficient condition for*
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A(u_1, u_2, \cdots)$$
is
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C,$$
for all n, and in the case of each n, for every possible choice of σ_k, where σ_k is any number such that $|\sigma_k| = 1$, and C is a constant.

We will assume
$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C$$
violated and show that a convergent sequence (u_n) exists such that
$$z_n = \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| < A$$
is not true for any value of A.

Choose n_1 and $\sigma_k^{(1)}$, $1 \leqq k \leqq n_1$, such that $|\sigma_k^{(1)}| = 1$ and

$$\sum_{l=1}^{n_1} \Big| \sum_{k=1}^{n_1} a_{n_1 kl} \sigma_k^{(1)} \Big| > 1.$$

Define

$$u_k = \sigma_k^{(1)}, \qquad 1 \leqq k \leqq n_1.$$

Then

$$z_{n_1} = \sum_{l=1}^{n_1} \Big| \sum_{k=1}^{n_1} a_{n_1 kl} u_k \Big| > 1.$$

By lemma 1

$$\sum_{l=1}^{n} | a_{nkl} | < K.$$

In general, choose

$$n_p > n_{p-1} \quad \text{and} \quad \sigma_k^{(p)}, \qquad 1 \leqq k \leqq n_p,$$

such that $|\sigma_k^{(p)}| = 1$ and

$$\sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_p} a_{n_p kl} \sigma_k^{(p)} \Big| > p^2 + 2pn_{p-1}K.$$

Define

$$u_k = \sigma_k^{(p)}/p, \qquad n_{p-1} < k \leqq n_p.$$

Then

$$\begin{aligned} z_{n_p} &= \sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_{p-1}} a_{n_p kl}(u_k - \sigma_k^{(p)}/p) + \sum_{k=1}^{n_p} a_{n_p kl}(\sigma_k^{(p)}/p) \Big| \\ &\geqq (1/p) \sum_{l=1}^{n_p} \Big| \sum_{k=1}^{n_p} a_{n_p kl} \sigma_k^{(p)} \Big| - \sum_{k=1}^{n_{p-1}} \sum_{l=1}^{n_p} | a_{n_p kl} | \cdot 2 \\ &> (1/p)\,[p^2 + 2pn_{p-1}K] - 2n_{p-1}K \\ &> p. \end{aligned}$$

Since z_{n_p} can be made as great as we please while $u_n \to 0$, the condition is necessary.

To prove the sufficiency of the condition assume *

$$\sum_{l=1}^{n} \Big| \sum_{k=1}^{n} a_{nkl} \sigma_k \Big| < C.$$

Let (u_n) be a convergent sequence; then $|u_n| < U$. Define

$$\sigma_l = \operatorname{sgn} \sum_{k=1}^{n} a_{nkl} u_k, \quad \text{when} \quad \sum_{k=1}^{n} a_{nkl} u_k \neq 0,$$

* The assumption insures that

$$\sum_{k=1}^{n} \Big| \sum_{l=1}^{n} a_{nkl} \sigma_l \Big| < C,$$

as may be seen by interchanging the rôles of ρ and σ in the latter part of the proof of lemma 3.

$$\sigma_l = 1, \qquad \text{when} \quad \sum_{k=1}^{n} a_{nkl} u_k = 0.$$

Then

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} u_k \right| = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl} u_k \right) \sigma_l \leqq \sum_{k=1}^{n} | u_k | \; \left| \sum_{l=1}^{n} a_{nkl} \sigma_l \right|$$

$$< U \sum_{k=1}^{n} \left| \sum_{l=1}^{n} a_{nkl} \sigma_l \right| < UC.$$

LEMMA 3. *Condition* $\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C$ *of lemma 2 may be replaced by*

$$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl} \sigma_k \rho_l < C,$$

for all n, and in the case of each n, for every possible choice of σ_k *and* ρ_l, *where* $| \sigma_k | = 1$, $| \rho_l | = 1$, *and C is a constant.*

Assuming

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C,$$

we have

$$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl} \sigma_k \rho_l \leqq \sum_{l=1}^{n} \left| \rho_l \sum_{k=1}^{n} a_{nkl} \sigma_k \right| \leqq \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C.$$

Assuming

$$\sum_{l=1}^{n} \sum_{k=1}^{n} a_{nkl} \sigma_k \rho_l < C,$$

define

$$\rho_l = \operatorname{sgn} \sum_{k=1}^{n} a_{nkl} \sigma_k, \qquad \text{when} \quad \sum_{k=1}^{n} a_{nkl} \sigma_k \neq 0,$$

$$\rho_l = 1, \qquad \text{when} \quad \sum_{k=1}^{n} a_{nkl} \sigma_k = 0.$$

Then

$$\sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{nkl} \sigma_k \right) \rho_l = \sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} \sigma_k \right| < C.$$

The fifth condition in Theorem I′ follows from the seventh condition in Theorem I, and conversely. We may therefore omit the fifth condition in both theorems. Referring then to the results in the preceding lemmas, we may state Theorems I and I′ together in the following form.

THEOREM II. *A necessary and sufficient condition that the transformation T be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each k and l,* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each l,* $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each k,* $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} \sigma_k \rho_l < C$,

for all n, and in the case of each n, for every possible choice of σ_k *and* ρ_l, *where* $|\sigma_k| = 1$, $|\rho_l| = 1$, *and* C *is a constant; and one or the other of*

$$\begin{cases} \sum_{k=1}^{n} |a_{nkl}| < L & \text{for all } l \text{ and } n, \\ \sum_{l=1}^{n} |a_{nkl}| < K & \text{for all } k \text{ and } n, \end{cases}$$

where K *and* L *are constants.*

2. *A necessary condition for regularity.* We notice that the last three conditions of Theorems I and I′ will follow from $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C$ for all n, where C is a constant. We can therefore state the following theorem.

THEOREM III. *A sufficient condition that the transformation* T *be regular is that* $\lim_{n\to\infty} a_{nkl} = 0$ *for each* k *and* l, $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nkl} = 0$ *for each* l, $\lim_{n\to\infty} \sum_{l=1}^{n} a_{nkl} = 0$ *for each* k, $\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{nkl} = 1$, $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C$ for all n, *where* C *is a constant.*

We have not been able to show that the last condition of this theorem is necessary for regularity of T. The following theorem indicates to what extent we have been able to state a necessary condition in terms of $|a_{nkl}|$.

THEOREM IV. *A necessary condition that the transformation* T *be regular is that* $\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < C(n)^{1/2}$ *for all* n, *where* C *is a constant.*

The proof of this theorem will be given at the end of this section after we have first stated certain lemmas. It is to be noted here that the last condition of Theorem II shows that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} |a_{nkl}| < Cn,$$

where C is a constant, is certainly necessary.

LEMMA 4. *Let* $G(x_1, \cdots, x_n)$ *denote the greatest of* $x_1, x_2, \cdots, x_n$, $x_1, x_2, \cdots, x_n$ *being real and non-negative. Then* $G(x_1, \cdots, x_n)$ *is continuous.*

A similar statement holds for $L(x_1, \cdots, x_n)$, the least of the numbers $x_1, x_2, \cdots, x_n$.

In the following lemmas we shall introduce the notation

$$\sum{}^* \mid \pm a_1 \pm a_2 \pm \cdots \pm a_n \mid$$

to denote the sum of absolute values for every possible combination of signs. For example

$$\sum{}^* \mid \pm a_1 \pm a_2 \mid = \mid a_1 + a_2 \mid + \mid a_1 - a_2 \mid + \mid -a_1 + a_2 \mid + \mid -a_1 - a_2 \mid .$$

LEMMA 5. *If any two a's in* $\sum^* \mid \pm a_1 \pm \cdots \pm a_n \mid$ *are unequal, the new summation in which each of these a's is replaced by their arithmetic mean, has a value not greater than the original sum.*

LEMMA 6. *The function,*

$$\frac{\sum^* \mid \pm a_1 \pm \cdots \pm a_n \mid}{\sum_{k=1}^{n} \mid a_k \mid},$$

of n real variables, $a_1, a_2, \cdots, a_n$, *not all zero, takes on its minimum value when all of the a's are equal.*

Let us investigate this minimum value by putting

$$a_k = 1, \qquad (k = 1, 2, \cdots, n).$$

Call

$$A_n = \sum{}^* \mid \pm 1 \pm 1 \pm \cdots \pm 1 \mid ,$$
$$S_n = \mid 1 \mid + \mid 1 \mid + \cdots + \mid 1 \mid = n,$$
$$Q_n = A_n / S_n .$$

Then by lemma 6

$$\frac{\sum^* \mid \pm a_1 \pm \cdots \pm a_n \mid}{\sum_{k=1}^{n} \mid a_k \mid} \geqq Q_n$$

whatever the values of the a's may be. It is not difficult to show that

$$Q_n = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \qquad n \text{ even};$$

$$Q_n = \frac{2(n-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}, \qquad n \text{ odd};$$

and as a result †

$$Q_n \sim \frac{2^{n+1}}{(2n\pi)^{1/2}} .$$

† $\alpha_n \sim \beta_n$ means $\lim_{n\to\infty} (\alpha_n/\beta_n) = 1$.

LEMMA 7. *When the transformation T is regular and a_{nkl} is complex, $a_{nkl} = \alpha_{nkl} + i\beta_{nkl}$, then it is necessary that*

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} \alpha_{nkl} x_k \right| < C,$$

and

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} \beta_{nkl} x_k \right| < C,$$

for all n, and in the case of each n, for every possible choice of $x_k = \pm 1$ where C is a constant.

We are now ready to prove Theorem IV.

(a) Let a_{nkl} be real. By lemma 2, calling $\sigma_k = \pm 1$, we have

$$\sum_{l=1}^{n} \left| \sum_{k=1}^{n} a_{nkl} x_k \right| < C,$$

for all n, and in the case of each n, for every possible choice of signs, $x_k = \pm 1$, where C is a constant. It follows immediately that

$$\Sigma^* [\, | \pm a_{n11} \pm \cdots \pm a_{nn1} | + \cdots + | \pm a_{n1n} \pm \cdots \pm a_{nnn} | \,] < 2^n C,$$

for there are evidently 2^n choices of signs.

But $$\sum_{k=1}^{n} | a_{nkl} | \leqq (1/Q_n) \Sigma^* | \pm a_{n1l} \pm \cdots \pm a_{nnl} |$$

for each l; therefore

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | \leqq (1/Q_n) \sum_{l=1}^{n} \Sigma^* | \pm a_{n1l} \pm \cdots \pm a_{nnl} | < 2^n C/Q_n.$$

Referring to the formula for Q_n, we have immediately

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | \sim \frac{2^n C}{2^{n+1}/(2n\pi)^{1/2}} = C'(n)^{1/2},$$

where C' is a constant independent of n.

(b) By lemma 7, and as a result of part (a) just proved, it follows immediately that when $a_{nkl} = \alpha_{nkl} + i\beta_{nkl}$, then

$$\sum_{l=1}^{n} \sum_{k=1}^{n} | a_{nkl} | < C'(n)^{1/2}.$$

3. *Application to the Cauchy Product.* In this section we shall give certain necessary and sufficient conditions under which a transformation of the type

$$S: \qquad y_n = \sum_{k=1}^{n} a_{nk} x_k, \qquad\qquad a_{nk} \text{ constant,}$$

will correctly evaluate the Cauchy product of two series each of which is Cèsaro summable. It will at times be convenient to write

$$a_{nk} = 0, \qquad\qquad (k > n).$$

Let $\sum u_n$ and $\sum v_n$ be two series, and let us call

$$U_n = \sum_{k=1}^{n} u_k, \qquad V_n = \sum_{k=1}^{n} v_k.$$

We shall write

$$U_0 = 0, \qquad V_0 = 0.$$

The Cauchy product of the two series is $\sum_{k=1}^{\infty} w_k$, where

$$w_k = \sum_{l=1}^{k} u_l v_{k-l+1}.$$

Let us write

$$W_n = \sum_{l=1}^{n} w_k = \sum_{l=1}^{n} u_l V_{n-l+1}.$$

The transformation S applied to W_n gives

$$y_n = \sum_{l=1}^{n} \sum_{k=1}^{n} (a_{n,k+l-1} - a_{n,k+l}) U_l V_k.$$

This is of the form T where $a_{nkl} = a_{n,k+l-1} - a_{n,k+l}$.

In defining Cèsaro summability of non-integral orders for the series $\sum u_n$, Chapman * considers

$$C_r: \qquad \phi_k = U_k^{(r)} / A_k^{(r)},$$

where

$$U_k^{(r)} = \sum_{l=1}^{k} \binom{r}{k-l} U_l,$$

and

$$A_k^{(r)} = \sum_{l=1}^{k} \binom{r}{k-l} = \binom{r+k-1}{k-1}.$$

Whenever

$$\lim_{k \to \infty} \phi_k = \lambda$$

the series $\sum u_n$ is said to be *summable* C_r to the value λ.

It is easy to obtain the inverse transformation

$$C_r^{-1}: \qquad U_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{r}{l-1} \binom{r+k-l}{k-l} \phi_{k-l+1}.$$

---

* *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 9 (1910), p. 369.

LEMMA 8. *If transformation S evaluates to uv the Cauchy product of every two series summable to values u and v by C_r and C_s respectively, where $r > 0$, $s > 0$, then S is regular.*

Let $\sum u_n$ be any convergent series; it is summable C_r, $r > 0$. Let

$$\sum v_n = 1 + 0 + 0 + \cdots,$$

which is summable C_s. The Cauchy product of these series is $\sum u_n$. Hence S must be regular since it evaluates $\sum u_n$ correctly.

Let $\sum u_n$ be summable C_r, $r > 0$, to the value u, and let $\sum v_n$ be summable C_s, $s > 0$, to the value v. In the notation previously explained write

$$C_r: \quad \phi_k = U_k^{(r)}/A_k^{(r)},$$

$$C_s: \quad \psi_k = V_k^{(s)}/A_k^{(s)},$$

$$C_r^{-1}: \quad U_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{r}{l-1} \binom{r+k-l}{k-l} \phi_{k-l+1},$$

$$C_s^{-1}: \quad V_k = \sum_{l=1}^{k} (-1)^{l-1} \binom{s}{l-1} \binom{s+k-l}{k-l} \psi_{k-l+1}.$$

Applying transformation S to the Cauchy product we have *

$$y_n = \sum_{k=1}^{n} \sum_{l=1}^{n} (a_{n,k+l-1} - a_{n,k+l}) U_l V_k$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \phi_l \psi_k.$$

This latter form of y_n is obtained by substituting the values of U_l and V_k as above and carrying out the necessary reductions. We have now a case of transformation T wherein

$$a_{nkl} = \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1}.$$

The necessary and sufficient conditions that the transformation S evaluate correctly the Cauchy product of $\sum u_n$ and $\sum v_n$ are obtained by using this value of a_{nkl} in the conditions of Theorem II. The results thus obtained may be simplified † to the form given in the following theorem.

---

* $\Delta^q x_p = \sum_{l=0}^{\infty} (-1)^l \binom{q}{l} x_{p+l}$, where $x_k = 0$ for sufficiently great values of k.

† For example

$$\sum_{k=1}^{n} \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} = \binom{r+l-1}{l-1} \Delta^r a_{nl},$$

THEOREM V. *A necessary and sufficient condition that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r, $r > 0$, and C_s, $s > 0$, to values u and v respectively, is that S be regular,*

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1}\sigma_k\rho_l < C,$$

for all n, and in the case of each n, for every possible choice of σ_k and ρ_l, where $|\sigma_k| = 1$, $|\rho_l| = 1$, and C is a constant; and one or the other of

$$\left\{\begin{array}{l} \displaystyle\sum_{k=1}^{n}\left|\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1}\right| < C, \text{ for all } n \text{ and } l, \\ \displaystyle\sum_{l=1}^{n}\left|\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1}\right| < C, \text{ for all } n \text{ and } k. \end{array}\right.$$

By Theorem III we may evidently state a sufficient condition thus:

THEOREM VI. *A sufficient condition that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r, $r > 0$, and C_s, $s > 0$, to values u and v respectively, is that S be regular,*

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left|\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1}\right| < C,$$

for all n, where C is a constant.

Also, by Theorem IV, we may state a further necessary condition as follows:

THEOREM VII. *A necessary condition, (in addition to those of Theorem V), that the transformation S evaluate to uv, the Cauchy product of every two series summable C_r and C_s to values u and v respectively is that*

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left|\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1}\right| < C(n)^{1/2},$$

for all values of n, where C is a constant.

4. *Conclusion.* There is still the possibility that the last condition of Theorem III may be necessary for regularity of T. If this is not the case it may happen that when this condition is replaced by that of Theorem IV

---

whence

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\binom{r+l-1}{l-1}\binom{s+k-1}{k-1}\Delta^{r+s+1}a_{n,k+l-1} = \sum_{k=1}^{\infty}a_{nk}.$$

Also $\lim_{n\to\infty}\Delta^r a_{nl} = 0$, $r > 0$, for each l. Lemma 8 reduces the first four conditions obtained to the condition of regularity of S.

the set of conditions will then be necessary and sufficient. Both of these questions remain open.

In this connection it is interesting to note that regularity of S together with

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \binom{r+l-1}{l-1} \binom{s+k-1}{k-1} \Delta^{r+s+1} a_{n,k+l-1} \right| < C$$

is a necessary and sufficient condition that S include C_{r+s+1}, when $r > 0$, $s > 0$.

If it should happen that the last condition of Theorem III be necessary, it would follow that a transformation S, evaluating correctly the Cauchy product of every two series summable C_r and C_s respectively, must include C_{r+s+1}. This would show the connection between our theory and Chapman's result that the Cauchy product of two series summable C_r and C_s respectively, is always summable C_{r+s+1}.

UNIVERSITY OF VERMONT.

ON SOME GENERAL COMMUTATION FORMULAS.*

By Neal H. McCoy.†

Let x_i $(i = 1, 2, \cdots, n)$ be a class of elements satisfying the ordinary laws of algebra with the exception that multiplication is not necessarily commutative.‡ In the first part of this paper we give two identities for the commutator, $fg - gf$, where f is an arbitrary polynomial in the x's and g is a polynomial of the form $\sum a_{m_1 m_2 \ldots m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, each with real or complex coefficients. These identities are obtained in terms of expressions of the form $\phi x_i - x_i \phi$ where ϕ is a function of the x's, and are generalizations of some formulas given by Wentzel.§

As the first application of these identities we consider a special non-commutative algebra which arises in quantum mechanics.¶ For a single pair of quantum "variables," p and q, the properties of the algebra are determined by the fundamental rule,

$$pq - qp = cI, \tag{1}$$

where c is a real or complex number. As is well known these variables may be interpreted either as infinite matrices, in which case I in relation (1) indicates the unit matrix, or they may be certain operators and in this case I represents the unit operator. The results which we obtain are independent of the particular interpretation to be placed on the variables. We shall omit the symbol "I" in what follows as there can be no confusion.

If ϕ is a polynomial in p and q we define,

$$\phi p - p\phi = - c\partial\phi/\partial q, \qquad \phi q - q\phi = c\partial\phi/\partial p, \tag{2}$$

from which it follows that the usual formulas for differentiating polynomials hold.‖ By means of the general identities discussed above we can obtain two

* Presented to the American Mathematical Society, September 11, 1930.

† National Research Fellow.

‡ By this statement we shall understand that the class of polynomials in the x's with real or complex coefficients constitute a non-commutative domain of integrity.

§ *Zeitschrift für Physik*, Vol. 37 (1926), p. 85.

¶ For references to this algebra see a previous paper, "On Commutation Formulas in the Algebra of Quantum Mechanics," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 793-806.

‖ Care must be taken to preserve the order of factors. That is,

$$\partial f\phi/\partial p = f\partial\phi/\partial p + (\partial f/\partial p)\phi.$$

formulas for the commutator of any two polynomials in p and q in terms of the various derivatives of these polynomials. These formulas are in a different form from those given previously. Corresponding results are obtained for the case of polynomials in n pairs of quantum variables, these being subject to the set of relations

$$(3) \qquad p_r q_s - q_s p_r = c\delta_{rs}, \qquad p_r p_s - p_s p_r = 0, \qquad q_r q_s - q_s q_r = 0.$$

We shall also give some commutation relations for functions of three variables, α, β, γ, which satisfy the conditions,

$$(4) \qquad \alpha\beta - \beta\alpha = c\gamma, \qquad \gamma\alpha - \alpha\gamma = c\beta, \qquad \beta\gamma - \gamma\beta = c\alpha.$$

These are the relations which hold for the components of angular momentum in quantum mechanics. As a special case it may be verified that the relations (4) are satisfied if we take

$$\alpha = q_2 p_3 - q_3 p_2, \qquad \beta = q_1 p_3 - q_3 p_1, \qquad \gamma = q_1 p_2 - q_2 p_1,$$

where p_1, p_2, p_3, q_1, q_2, q_3 are subject to the relations (3). As before, the results obtained depend only upon the fact that the variables satisfy relation (4) and not upon any special interpretation of the variables.

1. *Two general identities.* Let x_i $(i = 1, 2, \cdots, n)$ be the elements considered. These elements are assumed to satisfy the ordinary laws of algebra with the exception that multiplication is non-commutative. As a special case we may at any time let certain of these x's be identical.

Let f be any polynomial in the x's with real or complex coefficients. We define operators D_{x_i} as follows:

$$(5) \qquad f x_i - x_i f = c D_{x_i} f, \qquad (i = 1, 2, \cdots, n)$$

where c is a fixed real or complex number. As a consequence of the definition the following properties of the operators D_{x_i} may be deduced:

$$(6) \qquad D_{x_j}\phi = 0 \text{ if } \phi \text{ is a function of } x_j \text{ only},$$

$$(7) \qquad D_{x_i}(f \pm g) = D_{x_i} f \pm D_{x_i} g,$$

$$(8) \qquad D_{x_i} fg = f D_{x_i} g + (D_{x_i} f) g.$$

It is clear from these properties that if $D_{x_i} x_j$ $(i, j = 1, 2, \cdots, n)$ are given then $D_{x_i} f$ is uniquely determined, where f is any polynomial in the x's.

We shall understand by $D_{x_i} D_{x_j} f$ the expression obtained by first operating with D_{x_j} on f and then operating upon the resulting expression with D_{x_i}. That is

$$c^2 D_{x_i} D_{x_j} f = (f x_j - x_j f) x_i - x_i (f x_j - x_j f).$$

In like manner we define $D_{x_i}{}^2$, $D_{x_i}{}^3$, $D_{x_i}D_{x_j}D_{x_k}$ and so on. For convenience we let $D_{x_i}{}^0 f$ denote f itself, that is $D_{x_i}{}^0$ is the unit operator.

In what follows we let f represent an arbitrary polynomial in the x's. We now prove that *

$$fx_i{}^n - x_i{}^n f = \sum_{s=1}^{n} c^s \binom{n}{s} x_i{}^{n-s} D_{x_i}{}^s f. \tag{9}$$

This relation is seen to reduce to (5) if $n = 1$ and hence is true for this case by definition. We accordingly assume it to be valid for a given n and show that it holds also for $n+1$. We have

$$\begin{aligned} fx_i{}^{n+1} - x_i{}^{n+1} f &= (fx_i{}^n - x_i{}^n f)x_i + x_i{}^n(fx_i - x_i f) \\ &= \left[\sum_{s=1}^{n} c^s \binom{n}{s} x_i{}^{n-s} D_{x_i}{}^s f\right] x_i + c x_i{}^n D_{x_i} f. \end{aligned}$$

But $(D_{x_i}{}^s f)x_i - x_i(D_{x_i}{}^s f) = cD_{x_i}{}^{s+1} f$, by (5) and we get

$$fx_i{}^{n+1} - x_i{}^{n+1} f = \sum_{s=1}^{n} c^s \binom{n}{s} x_i{}^{n-s+1} D_{x_i}{}^s f + \sum_{s=1}^{n} c^{s+1} \binom{n}{s} x_i{}^{n-s} D_{x_i}{}^{s+1} f + c x_i{}^n D_{x_i} f.$$

By making use of the fact that

$$\binom{n}{s} + \binom{n}{s-1} = \binom{n+1}{s}, \tag{10}$$

it is easily verified that this is relation (9) with n replaced by $n+1$ which completes the proof of this formula. As a generalization of this result we have the theorem:

THEOREM I. *Let x_i $(i = 1, 2, \cdots, n)$ be any class of distinct elements satisfying the usual laws of algebra except that multiplication is not necessarily commutative. Let f be any polynomial in these x's and g a polynomial of the form $\sum a_{m_1 m_2 \ldots m_n} x_1{}^{m_1} x_2{}^{m_2} \cdots x_n{}^{m_n}$, each with real or complex coefficients. If the operators D_{x_i} $(i = 1, 2, \cdots, n)$ are defined by relation* (5) *then* (a)

$$fg - gf = \sum_{s=1} c^s \sum_{i_1+i_2+\ldots+i_n=s} \frac{1}{i_1!\, i_2! \cdots i_n!} \frac{\partial^s g}{\partial x_1{}^{i_1} \partial x_2{}^{i_2} \cdots \partial x_n{}^{i_n}} \times D_{x_n}{}^{i_n} \cdots D_{x_2}{}^{i_2} D_{x_1}{}^{i_1} f, \tag{11}$$

and (b),

$$fg - gf = -\sum_{s=1} (-c)^s \sum_{i_1+i_2+\ldots+i_n=s} \frac{1}{i_1!\, i_2! \cdots i_n!} (D_{x_1}{}^{i_1} D_{x_2}{}^{i_2} \cdots D_{x_n}{}^{i_n} f) \frac{\partial^s g}{\partial x_1{}^{i_1} \partial x_2{}^{i_2} \cdots \partial x_n{}^{i_n}}, \tag{12}$$

the sum in each case being taken over all non-vanishing terms.

---

* See Wentzel, *loc. cit.* The proof given here is essentially that given by Wentzel.

The derivatives used in these formulas are to be taken according to the usual rules for differentiating polynomials. We consider first the proof of part (a) of the theorem. Clearly if formula (11) is true for two functions g_1 and g_2 of the type there considered it will also be true for their sum. It is then sufficient to establish it for the case of a single term. We shall show that,

$$(13)\quad fx_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} - x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}f$$
$$= \sum_{s=1} c^s \sum_{i_1+i_2+\cdots+i_n=s} \binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n} x_1^{m_1-i_1}x_2^{m_2-i_2}\cdots x_n^{m_n-i_n} D_{x_n}^{i_n}\cdots D_{x_2}^{i_2}D_{x_1}^{i_1}f.$$

When written in this form the result remains true if the x's are not all distinct. For example we may have $x_1 = x_3 = x_4$, $x_2 = x_5$ and so on. The proof of relation (13) is by induction. It is true by equation (9) for the special case where all the m_i but one are zero and f is any polynomial. We assume then that the relation gives a true expression for $fx_i^n x_j^m - x_i^n x_j^m f$, and show that it remains true if n is replaced by $n+1$.

For convenience let $g' = x_i^n x_j^m$ and let

$$H^{(s)}(f, g') = \sum_{i_1+i_2=s} \binom{n}{i_1}\binom{m}{i_2} x_i^{n-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f.$$

Then equation (13) with g' in place of g can be written in the form,

$$(13')\qquad fg' - g'f = \sum_{s=1} c^s H^{(s)}(f, g'),$$

and we wish to show that this remains valid if g' is replaced by $x_i g'$. Now

$$H^{(s)}(f, x_i g') = \sum_{i_1+i_2=s} \binom{n+1}{i_1}\binom{m}{i_2} x_i^{n+1-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f$$
$$= x_i H^{(s)}(f, g') + \sum_{i_1+i_2=s} \binom{n}{i_1-1}\binom{m}{i_2} x_i^{n+1-i_1}x_j^{m-i_2}D_{x_j}^{i_2}D_{x_i}^{i_1}f$$

by formula (10). From this we find that

$$H^{(s)}(f, x_i g') = x_i H^{(s)}(f, g') + H^{(s-1)}(D_{x_i}f, g').$$

Hence we wish to verify that

$$fx_i g' - x_i g' f = x_i \sum_{s=1} c^s H^{(s)}(f, g') + \sum_{s=1} c^s H^{(s-1)}(D_{x_i}f, g').$$

But we have

$$fx_i g' - x_i g' f = x_i(fg' - g'f) + (fx_i - x_i f)g'$$
$$= x_i \sum_{s=1} c^s H^{(s)}(f, g') + c(D_{x_i}f)g'.$$

Hence we only need to show that

$$(D_{x_i}f)g' = \sum_{s=1} c^{s-1}H^{(s-1)}(D_{x_i}f, g')$$

or

$$(D_{x_i}f)g' - g'(D_{x_i}f) = \sum_{s=1} c^s H^{(s)}(D_{x_i}f, g')$$

which is true as we have made no restrictions on the polynomial f and relation (13′) thus remains true if f is replaced by $D_{x_i}f$. By going through the argument just given for the case $n = 0$ it is seen to be valid for that case also. Hence we have by induction that relation (13) is true if two of the m_i are different from zero and the others vanish. A repetition of this argument proves the general case.

In order to prove relation (12) we need to show that

$$(14)\quad fx_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} - x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}f$$
$$= -\sum_{s=1}(-c)^s \sum_{i_1+i_2+\cdots+i_n=s} \binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n}(D_{x_1}^{i_1}D_{x_2}^{i_2}\cdots D_{x_n}^{i_n}f)$$
$$\times\, x_1^{m_1-i_1}x_2^{m_2-i_2}\cdots x_n^{m_n-i_n}.$$

We shall derive this identity from relation (13) which has been established.

The identity (13) is a formal identity, that is if the D's are replaced by their expressions from (5) everything will cancel out. Hence if we apply any kind of transformation to every term the result will still be an identity. Suppose then in (13) we reverse the order of all factors and change the sign of c. If ϕ is any function of the x's denote by $\bar{\phi}$ the function obtained from ϕ by this transformation. For example if

$$\phi = x_1x_3^2x_1x_2 + cx_1x_2x_3^2x_1^3,$$

then

$$\bar{\phi} = x_2x_1x_3^2x_1 - cx_1^3x_3^2x_2x_1.$$

From (5) it is seen that $\overline{D_{x_i}f} = D_{x_i}\bar{f}$, for

$$x_i\bar{f} - \bar{f}x_i = -c\overline{D_{x_i}f} = -cD_{x_i}\bar{f}.$$

Hence the result of applying this transformation to the identity (13) is to obtain the identity,

$$x_n^{m_n}\cdots x_2^{m_2}x_1^{m_1}\bar{f} - \bar{f}x_n^{m_n}\cdots x_2^{m_2}x_1^{m_1}$$
$$= \sum_{s=1}(-c)^s \sum_{i_1+i_2+\cdots+i_n=s} \binom{m_1}{i_1}\binom{m_2}{i_2}\cdots\binom{m_n}{i_n}(D_{x_n}^{i_n}\cdots D_{x_2}^{i_2}D_{x_1}^{i_1}\bar{f})$$
$$\times\, x_n^{m_n-i_n}\cdots x_2^{m_2-i_2}x_1^{m_1-i_1}.$$

This becomes relation (14) if we replace the general polynomial $\bar{f}$ by f and in the subscripts replace n by 1, $n-1$ by 2 and so on, which obviously does not affect the truth of the relation. This completes the proof of Theorem I.

It is interesting to note that relation (11), for example, may be written symbolically as

$$fg - gf = \sum_{s=1} (c^s/s!)[(\partial g/\partial x_1)D_{x_1}f + (\partial g/\partial x_2)D_{x_2}f + \cdots + (\partial g/\partial x_n)D_{x_n}f]^s,$$

and similarly for relation (12).

2. *The algebra of quantum mechanics.* We now take up an application of the identities obtained above to a special algebra arising in quantum mechanics. Consider first a single pair of variables, p and q, satisfying the relation,

$$pq - qp = c, \tag{15}$$

where c is a real or complex number and thus commutes with any function of p and q.

It is found that in this case

$$D_pf = -\partial f/\partial q, \qquad D_qf = \partial f/\partial p,$$

by equations (2). We have further that

$$D_pD_qf = D_qD_pf, \tag{16}$$

a result which will greatly simplify the formulas obtained. The result (16) may be verified by substituting for D_p and D_q from (2). We have by making this substitution,

$$(fq - qf)p - p(fq - qf) = (fp - pf)q - q(fp - pf),$$

which may be easily verified by use of relation (15).

Let f be an arbitrary polynomial in p and q and let g take, for example, the special form, $p^{m_1}q^{m_2}p^{m_3}q^{m_4}$. We have then from formula (13),

$$fg - gf = \sum_{s=1} c^s \sum_{i_1+i_2+i_3+i_4=s} (-1)^{i_1+i_3} \binom{m_1}{i_1}\binom{m_2}{i_2}\binom{m_3}{i_3}\binom{m_4}{i_4} \times p^{m_1-i_1}q^{m_2-i_2}p^{m_3-i_3}q^{m_4-i_4} \frac{\partial^s f}{\partial p^{i_2+i_4}\partial q^{i_1+i_3}}.$$

But by Leibnitz' formula for the k_1-th derivative of a product it is easily seen that,

$$\sum_{i_1+i_3=k_1} \binom{m_1}{i_1}\binom{m_3}{i_3} p^{m_1-i_1}q^{m_2}p^{m_3-i_3}q^{m_4} = \frac{1}{k_1!}\,\frac{\partial^{k_1} g}{\partial p^{k_1}},$$

and in general

$$\sum_{i_1+i_3=k_1}\ \sum_{i_2+i_4=k_2} \binom{m_1}{i_1}\binom{m_2}{i_2}\binom{m_3}{i_3}\binom{m_4}{i_4} p^{m_1-i_1}q^{m_2-i_2}p^{m_3-i_3}q^{m_4-i_4} = \frac{1}{k_1!\,k_2!}\ \frac{\partial^{k_1+k_2} g}{\partial p^{k_1}\partial q^{k_2}}.$$

We have thus verified a special case of part (a) of the theorem:

THEOREM II. *If f and g are arbitrary polynomials in p and q, then* (a)*

$$fg - gf = \sum_{s=1} c^s \sum_{k_1+k_2=s} \frac{(-1)^{k_1}}{k_1!\,k_2!} \frac{\partial^s g}{\partial p^{k_1} \partial q^{k_2}} \frac{\partial^s f}{\partial q^{k_1} \partial p^{k_2}}, \tag{17}$$

and (b),

$$fg - gf = -\sum_{s=1} c^s \sum_{k_1+k_2=s} \frac{(-1)^{k_2}}{k_1!\,k_2!} \frac{\partial^s f}{\partial q^{k_1} \partial p^{k_2}} \frac{\partial^s g}{\partial p^{k_1} \partial q^{k_2}}, \tag{18}$$

the sum in each case being taken over all non-null derivatives of f and g.

The first part of this theorem is seen by the argument just given to be valid if $g = p^{m_1} q^{m_2} p^{m_3} q^{m_4}$. The argument is however quite general and will apply to any term of a polynomial by simply using the formula for the n-th derivative of a product of any number of functions. Further if it is true for any two polynomials it is true for their sum which completes the proof of formula (17). Formula (18) is obtained in like manner from relation (14).

Let us consider polynomials in the $2n$ variables $p_1, q_1, \cdots, p_n, q_n$ satisfying the relations (3). In this case we have

$$p_r f - f p_r = c\partial f / \partial q_r = -cD_{p_r} f$$

and

$$f q_r - q_r f = c \frac{\partial f}{\partial p_r} = cD_{q_r} f.$$

It is again easily verified that $D_{p_r} D_{q_s} f = D_{q_s} D_{p_r} f$ $(r, s = 1, 2, \cdots, n)$. The following theorem may therefore be proved in a way similar to the proof of Theorem II.

THEOREM III. *If f and g are arbitrary polynomials in the $2n$ variables, $p_1, q_1, p_2, q_2, \cdots, p_n, q_n$ satisfying the relations* (3), *then* (a)

$$\begin{aligned} (19)\quad & fg - gf \\ &= \sum_{s=1} c^s \sum_{k_1+k_2+\cdots+k_{2n}=s} \frac{(-1)^{k_1+k_3+\cdots+k_{2n-1}}}{k_1!\,k_2!\cdots k_{2n}} \frac{\partial^s g}{\partial p_1^{k_1} \partial q_1^{k_2} \cdots \partial p_n^{k_{2n-1}} \partial q_n^{k_{2n}}} \\ &\qquad \times \frac{\partial^s f}{\partial q_1^{k_1} \partial p_1^{k_2} \cdots \partial q_n^{k_{2n-1}} \partial p_n^{k_{2n}}}, \end{aligned}$$

and (b)

---

* In the previous paper referred to above the following expressions were obtained:

$$fg - gf = \sum_{s=1} \frac{c^s}{s!} \left[\frac{\partial^s g}{\partial q^s} \frac{\partial^s f}{\partial p^s} - \frac{\partial^s f}{\partial q^s} \frac{\partial^s g}{\partial p^s} \right]$$

and

$$fg - gf = \sum_{s=1} \frac{(-c)^s}{s!} \left[\frac{\partial^s g}{\partial p^s} \frac{\partial^s f}{\partial q^s} - \frac{\partial^s f}{\partial p^s} \frac{\partial^s g}{\partial q^s} \right].$$

We thus have four different forms for the commutator of any two polynomials in p and q

(20) $fg - gf$

$$= -\sum_{s=1} c^s \sum_{k_1+k_2+\cdots+k_{2n}=s} \frac{(-1)^{k_2+k_4+\cdots+k_{2n}}}{k_1!\,k_2!\cdots k_{2n}!} \frac{\partial^s f}{\partial q_1^{k_1}\partial p_1^{k_2}\cdots\partial q_n^{k_{2n-1}}\partial p_n^{k_{2n}}} \times \frac{\partial^s g}{\partial p_1^{k_1}\partial q_1^{k_2}\cdots\partial p_n^{k_{2n-1}}\partial q_n^{k_{2n}}},$$

the sum in each case being taken over all non-null derivatives of f and g.

3. *Another special non-cummutative algebra.* Let us consider functions of three elements or variables, α, β, γ which are subject to the conditions,

(21) $$\alpha\beta - \beta\alpha = c\gamma, \qquad \gamma\alpha - \alpha\gamma = c\beta, \qquad \beta\gamma - \gamma\beta = c\alpha.$$

We prove first the following theorem:

THEOREM IV. *Any identity in α, β, γ remains an identity if the order of all factors is reversed and c is replaced by $-c$.*

An identity of the form $\phi(\alpha,\beta,\gamma) - \phi(\alpha,\beta,\gamma) = 0$, will be called a formal identity. We here consider identities which may be obtained from formal identities by a finite number of substitutions from (21), which is true for all polynomial identities. The theorem is clearly true for formal identities. Hence we need only to show that if it is true for a given identity it remains true after making any one of the substitutions of (21). If ϕ is any function of α, β, γ, c let $\bar{\phi}$ denote the function obtained from ϕ by reversing the order of all factors and changing the sign of c. Let $f = 0$ be an identity such that $\bar{f} = 0$ is also true. In any term of f replace, for example, $\alpha\beta$ by $\beta\alpha + c\gamma$ from the first of relations (21) and denote by $f' = 0$ the resulting identity. Now $\bar{f}'$ differs from $\bar{f}$ only in that we have replaced in one term $\beta\alpha$ by $\alpha\beta - c\gamma$ and these are equivalent. Hence we have also $\bar{f}' = 0$. A similar argument holds for any of the substitutions obtainable from (21). In thus building up a given identity from a formal identity the theorem is true at each step and hence for the final identity.

THEOREM V. *In any identity replace α, β, γ, c by α', β', γ', kc respectively, where α', β', γ' are obtained from α, β, γ by the non-singular transformation with real or complex coefficients*

$$\alpha' = a_{11}\alpha + a_{12}\beta + a_{13}\gamma, \quad \beta' = a_{21}\alpha + a_{22}\beta + a_{23}\gamma, \quad \gamma' = a_{31}\alpha + a_{32}\beta + a_{33}\gamma$$

of determinant Δ, and k is a real or complex number. The result will be another identity if and only if $a_{ij} = kA_{ij}$ where A_{ij} denotes the co-factor of a_{ij} in Δ.

We have

$$\begin{aligned}\alpha'\beta'-\beta'\alpha' &= (a_{11}a_{22}-a_{21}a_{12})(\alpha\beta-\beta\alpha)\\ &\quad -(a_{11}a_{23}-a_{21}a_{13})(\gamma\alpha-\alpha\gamma)+(a_{12}a_{23}-a_{22}a_{13})(\beta\gamma-\gamma\beta)\\ &= c\,[(a_{11}a_{22}-a_{21}a_{12})\gamma+(a_{21}a_{13}-a_{11}a_{23})\beta+(a_{12}a_{23}-a_{22}a_{13})\alpha]\\ &= ck(a_{33}\gamma+a_{32}\beta+a_{31}\alpha)=ck\gamma',\end{aligned}$$

if the conditions of the theorem are satisfied. In like manner we find that

$$\gamma'\alpha'-\alpha'\gamma'=ck\beta' \quad\text{and}\quad \beta'\gamma'-\gamma'\beta'=ck\alpha'.$$

Thus α', β', γ' satisfy the same identities as α, β, γ with c replaced by kc which proves that the condition of the theorem is sufficient. If the matrix of the transformation is not of the prescribed form the theorem will fail when applied to the fundamental identities (21). It is easily seen that if $a_{ij}=kA_{ij}$, then k must be a cube root of $1/\Delta$.

As special cases of this theorem we find that from any identity we may obtain another (a) by cyclic permutation of the letters α, β, γ or (b) by interchanging any two letters and changing the sign of c. These may be seen by considering the identities (21) directly.

It may be shown by induction that

$$D_\gamma\alpha^n=-\sum_{s=0}^{n-1}\alpha^{n-s-1}\beta\alpha^s, \quad\text{and}\quad D_\gamma\beta^n=\sum_{s=0}^{n-1}\beta^{n-s-1}\alpha\beta^s. \tag{22}$$

The other operators are obtained from these by cyclic permutation. Let us prove the second of equations (22). Assuming it holds for n we find:

$$\begin{aligned}cD_\gamma\beta^{n+1} &= \beta^{n+1}\gamma-\gamma\beta^{n+1}=\beta(\beta^n\gamma-\gamma\beta^n)+(\beta\gamma-\gamma\beta)\beta^n\\ &= c\,\{\sum_{s=0}^{n-1}\beta^{n-s}\alpha\beta^s+\alpha\beta^n\}=c\sum_{s=0}^{n}\beta^{n-s}\alpha\beta^s.\end{aligned}$$

But the relation is easily seen to be true for $n=1$ which completes the proof. The formulas (22) together with (6), (7) and (8) may be taken as the definition of $D_\gamma f$ where f is any polynomial in α, β, γ. Other expressions for these operators will be given later.

We find that $D_\alpha D_\beta f\neq D_\beta D_\alpha f$ but that

$$\begin{aligned}c^2(D_\alpha D_\beta f-D_\beta D_\alpha f) &= (f\beta-\beta f)\alpha-\alpha(f\beta-\beta f)\\ &\quad -(f\alpha-\alpha f)\beta+\beta(f\alpha-\alpha f)\\ &= f(\beta\alpha-\alpha\beta)+(\alpha\beta-\beta\alpha)f\\ &= -c(f\gamma-\gamma f)=-c^2D_\gamma f.\end{aligned}$$

In like manner each of the following relations may be verified:

$$(D_\alpha D_\beta-D_\beta D_\alpha)f=-D_\gamma f, \qquad (D_\gamma D_\alpha-D_\alpha D_\gamma)f=-D_\beta f,$$
$$(D_\beta D_\gamma-D_\gamma D_\beta)f=-D_\alpha f.$$

Thus the operators D_α, D_β, D_γ satisfy relations of the type (21) with c replaced by -1. It follows that corresponding to any identity expressed in terms of α, β, γ, c, there corresponds another in $D_\alpha, D_\beta, D_\gamma, -1$, which may be interpreted as operating on an arbitrary function of α, β, γ.

Two general commutation formulas for polynomials in these variables are given in the following theorem which is an immediate consequence of Theorem I. Other formulas may be obtained by applying the transformations of Theorems IV and V to those given here.

THEOREM VI. *If f is an arbitrary polynomial in α, β, γ; g is a polynomial of the form $\sum a_{lmn}\alpha^l\beta^m\gamma^n$, and the operators D are defined by* (22), (6), (7) *and* (8), *then* (a)

$$fg - gf = \sum_{s=1} c^s \sum_{i+j+k=s} \frac{1}{i!\,j!\,k!} \frac{\partial^s g}{\partial\alpha^i\partial\beta^j\partial\gamma^k} D_\gamma{}^k D_\beta{}^j D_\alpha{}^i f, \tag{23}$$

and (b)

$$fg - gf = -\sum_{s=1} (-c)^s \sum_{i+j+k=s} \frac{1}{i!\,j!\,k!} (D_\alpha{}^i D_\beta{}^j D_\gamma{}^k f) \frac{\partial^s g}{\partial\alpha^i\partial\beta^j\partial\gamma^k}. \tag{24}$$

As an interesting special case of formula (24) let $f = \gamma$. Then $D_\gamma{}^k\gamma = 0$ $(k > 0)$ and hence we only need to calculate $D_\alpha{}^i D_\beta{}^j\gamma$. The following table gives the values of this expression for $i, j = 0, 1, 2, \cdots, 5$.

$i \backslash j$	0	1	2	3	4	5
0	γ	$-\alpha$	$-\gamma$	α	γ	$-\alpha$
1	β	0	$-\beta$	0	β	0
2	$-\gamma$	0	γ	0	$-\gamma$	0
3	$-\beta$	0	β	0	$-\beta$	0
4	γ	0	$-\gamma$	0	γ	0
5	β	0	$-\beta$	0	β	0

$D_\alpha{}^i D_\beta{}^j\gamma$.

It is also seen that $D_\alpha{}^i D_\beta{}^j\gamma = D_\alpha{}^{i+4} D_\beta{}^j\gamma = D_\alpha{}^i D_\beta{}^{j+4}\gamma$ for $i, j > 0$. We thus find from (24),

$$\begin{aligned}\gamma g - g\gamma &= c(\beta\partial g/\partial\alpha - \alpha\partial g/\partial\beta) + (c^2/2!)(\gamma\partial^2 g/\partial\alpha^2 + \gamma\partial^2 g/\partial\beta^2) \\ &+ (c^3/3!)(-\beta\partial^3 g/\partial\alpha^3 - 3\beta\partial^3 g/\partial\alpha\partial\beta^2 + \alpha\partial^3 g/\partial\beta^3) \\ &+ (c^4/4!)(-\gamma\partial^4 g/\partial\alpha^4 - 6\gamma\partial^4 g/\partial\alpha^2\partial\beta^2 - \gamma\partial^4 f/\partial\beta^4) \\ &+ (c^5/5!)(\beta\partial^5 g/\partial\alpha^5 + 10\beta\partial^5 g/\partial\alpha^3\partial\beta^2 + 10\beta\partial^5 g/\partial\alpha\partial\beta^4 - \alpha\partial^5 g/\partial\beta^5) + \cdots.\end{aligned} \tag{25}$$

If ϕ is a polynomial in α alone we get as a special case from (25),

$$\begin{aligned}\gamma\phi - \phi\gamma &= \beta[c\partial\phi/\partial\alpha - (c^3/3!)\partial^3\phi/\partial\alpha^3 + (c^5/5!)\partial^5\phi/\partial\alpha^5 - \cdots] \\ &+ \gamma[(c^2/2!)\partial^2\phi/\partial\alpha^2 - (c^4/4!)\partial^4\phi/\partial\alpha^4 + (c^6/6!)\partial^6\phi/\partial\alpha^6 - \cdots].\end{aligned} \tag{26}$$

From this it follows that

$$-cD_\gamma\alpha^n = \sum_{s=1}(-1)^{s+1}c^{2s-1}\binom{n}{2s-1}\beta\alpha^{n-2s+1} + \sum_{s=1}(-1)^{s+1}c^{2s}\binom{n}{2s}\gamma\alpha^{n-2s}.$$

Now as in the ordinary case we have

$$\phi(\alpha + a) = \phi(\alpha) + a\partial\phi(\alpha)/\partial\alpha + (a^2/2!)\partial^2\phi(\alpha)/\partial\alpha^2 + \cdots$$

where a is a real or complex number. By making use of this fact it may be verified from (26) that if $i = (-1)^{1/2}$, then

$$\gamma\phi(\alpha) - \phi(\alpha)\gamma = (\beta/2i)\,[\phi(\alpha + ci) - \phi(\alpha - ci)] \\ \qquad -(\gamma/2)\,[\phi(\alpha + ci) + \phi(\alpha - ci) - 2\phi(\alpha)].$$

From each of the relations here deduced one may obtain others by cyclic permutation of the letters or by interchanging two letters and changing the sign of c.

PRINCETON UNIVERSITY.

2s.

pe

ic
ne